An Independent Axiom System for the Real Numbers

Greg Oman

UCCS
goman@uccs.edu

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$\square$

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Example

The property of commutativity of a group operation $*$ is independent from the usual axioms for a group since there exist both Abelian and non-Abelian groups (for example, $(\mathbb{Z}, +)$ and $(S_3, \circ)$).

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In fact, Cohen won a Fields Medal for this work (the only Fields Medal awarded to a logician to date).

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In this talk, we describe a categorical, independent axiom system for the ordered field of real numbers.

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Thus $a+b+a+b = a+a+b+b$. Canceling the first and last terms yields $b+a = a+b$. □

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- Moving toward a minimal set of axioms, consider the following system:

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Let us define a **complete ordered algebra** to be a 5-tuple $(F, +, \cdot, 0, <)$ consisting of a set F, operations $+$ and $\cdot$ on F, an element $0 \in F$, and a relation $<$ on F which satisfies the following axioms:

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(D) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in F$
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Let us define a **complete ordered algebra** to be a 5-tuple $(F, +, \cdot, 0, <)$ consisting of a set F, operations $+$ and $\cdot$ on F, an element $0 \in F$, and a relation $<$ on F which satisfies the following axioms:

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Surprisingly, these properties can actually be deduced as *theorems*, and need not be assumed as axioms.

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STEP 3: Define an operation $\circ$ on $\mathbb{R}$ by $x \circ y := f(f^{-1}(x) \odot f^{-1}(y))$. One then shows that $(A, +, \odot, 0, <) \cong (\mathbb{R}, +, \circ, 0, <) \cong (\mathbb{R}, +, \cdot, 0, <)$. $\square$

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Model: Let $(F, +, \cdot, 0, <) := (\{0\}, +, \cdot, 0, \varnothing)$. $\square$

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Hence we have established the following theorem:

Theorem

The axioms for a complete ordered algebra are categorical and independent, and the reals yield the unique model of the axioms up to isomorphism.