Groups where free subgroups are abundant

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2. $(\mathbb{R}, +)$ is a *completely metrizable* topological group under its usual topology (i.e., the topology is defined by a metric, which happens to be complete).
3. The group of all permutations $\mathrm{Sym}(\Omega)$ of an infinite set Ω is a topological group under the *function topology*, which has a subbasis of open sets of the form $\{f \in \mathrm{Sym}(\Omega) : f(\alpha) = \beta\}$ $(\alpha, \beta \in \Omega)$.

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4. If $\{G_i\}_{i\in I}$ is any collection of topological groups, then $\prod_{i\in I} G_i$ is a topological group under the *product topology*, which has a subbasis of open sets of the form $\prod_{i\in I} U_i$, where for some $j \in I$, $U_j \subseteq G_j$ is an open set, and $U_i = G_i$ for $i \neq j$.

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2. $(\mathbb{R}, +)$ is Polish, since $\mathbb{Q} \subseteq \mathbb{R}$ is dense.
3. $\mathrm{Sym}(\mathbb{Z}_+)$ is Polish. For all $f, g \in \mathrm{Sym}(\mathbb{Z}_+)$, define

$$d(f,g) = \begin{cases} 0 & \text{if } f = g \\ 2^{-n} & \text{if } f \neq g \end{cases}$$

   where $n \in \mathbb{Z}_+$ is the least number such that either $f(n) \neq g(n)$ or $f^{-1}(n) \neq g^{-1}(n)$. Then d is a complete metric which induces the function topology on $\mathrm{Sym}(\mathbb{Z}_+)$, and the (countable) subset of all permutations that move only finitely many points is dense.

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4. A countable direct product of Polish groups is Polish.

Definition

Let T be subset of a topological space.

1. T is called *nowhere dense* if its closure contains no open subsets.
2. T is called *comeagre* if it is the complement of a countable union of nowhere dense sets.

Theorem (Dixon, 1990)

Let $S = \mathrm{Sym}(\mathbb{Z}_+)$. Then the set

$$\{(g_1, \ldots, g_n) \in S^n : \{g_1, \ldots, g_n\} \text{ freely generates a free subgroup of } S\}$$

is comeagre in S^n for each integer $n \geq 2$.

Theorem

The following groups S satisfy the conclusion of Dixon's theorem.

1. $(\mathrm{Glass/McCleary/Rubin}, 1993)$ $\mathrm{Aut}(\Omega, \leq)$, for any countable highly homogeneous poset $(\Omega, \leq)$.
2. $(\mathrm{Gartside/Knight}, 2003)$ Any Polish oligomorphic group.
3. $(\mathrm{Bryant/Roman'kov}, 1998)$ $\mathrm{Aut}(G)$, for any relatively free Ω-algebra G of infinite rank, where Ω is an operator domain.
4. $(\mathrm{Bhattarcharjee}, 1995)$ An inverse limit of wreath products of nontrivial groups.
5. $(\mathrm{Gartside/Knight}, 2003)$ The absolute Galois group of the rational numbers.
6. $(\mathrm{Epstein/Gartside/Knight}, 2003)$ Any finite-dimensional connected non-solvable Lie group.

Definition

Let G be a Polish group. Then G is *almost free* if

$$\{(g_1, \dots, g_n) \in G^n : \{g_1, \dots, g_n\} \text{ freely generates a free subgroup of } G\}$$

is comeagre in G^n for each $n \geq 2$, and G is *almost countably free* if

$$\{(g_1, g_2, \dots) \in G^{\mathbb{N}} : \{g_1, g_2, \dots\} \text{ freely generates a free subgroup of } G\}$$

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Theorem (Gartside/Knight, 2003)

Let G be a non-discrete Polish group. Then the following are equivalent.

1. G is almost free.
2. G is almost countably free.
3. G contains a dense free subgroup of rank ≥ 2.

Theorem (Baire Category)

In a complete metric space, the intersection of a countable collection of open dense sets is dense; equivalently, a comeagre set must be dense.

Examples

1. Countable non-discrete groups (in particular, countable free groups) are not completely metrizable, and hence not Polish. (If such a group is completely metrizable and has no isolated points, then it can be written as a countable union of nowhere dense sets, namely the singleton sets, which contradicts the Baire Category Theorem. But, if some element is isolated, then the group must be discrete.)

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2. If $|\Omega| > \aleph_0$, then $\mathrm{Sym}(\Omega)$ is not metrizable, since it is not first-countable. (A topological space is *first-countable* if each point has a countable base for its system of neighborhoods. Every metric space is first-countable, since the open balls centered at a point p, of radii $1/n$ $(n \in \mathbb{Z}_+)$ form a countable base for p.)

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3. For any group G and any non-Polish group H, $G \times H$ is not Polish.

Definition (Gartside/Knight)

Let G be a Polish group. Then G is *almost free* if

$$\{(g_1, \ldots, g_n) \in G^n : \{g_1, \ldots, g_n\} \text{ freely generates a free subgroup of } G\}$$

is comeagre in G^n for each $n \geq 2$, and G is *almost countably free* if

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Remark

By the Baire Category Theorem, in a complete metric space a comeagre set is dense. But, *comeagre* is not a particularly useful notion in an arbitrary topological space.

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An infinite topological group G is *almost κ-free* if

$$G_\kappa = \{(g_i)_{i\in\kappa} \in G^\kappa : \{g_i\}_{i\in\kappa} \text{ freely generates a free subgroup of } G\}$$

is dense in G^κ, where $\kappa > 0$ a cardinal. Also, G is *almost free* if it is almost n-free for each $n \in \mathbb{Z}_+$, and G is *almost countably free* if it is almost $\aleph_0$-free.

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Lemma

Let G be a completely metrizable topological group and $1 \leq \kappa \leq \aleph_0$. Then G_κ is dense in G^κ if and only if G_κ is comeagre in G^κ.

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Proof.

The "if" direction follows from the Baire Category Theorem. For the converse, express $G^\kappa \setminus G_\kappa$ as the (countable) union of the closed sets $\{(g_i)_{i\in\kappa} \in G^\kappa : w(g_{i_1}, \ldots, g_{i_n}) = 1\}$, where $i_1, \ldots, i_n \in \kappa$ and w is a free word. If G_κ is dense in G^κ, then these sets are nowhere dense. □

Theorem (Gartside/Knight)

Let G be a non-discrete Polish group. Then the following are equivalent.

1. G is almost free.
2. G is almost countably free.
3. G contains a dense free subgroup of rank ≥ 2.

Proposition

Let G and H be topological groups, and let $\kappa > \lambda > 0$ be cardinals.

1. If G is almost κ-free, then it is almost λ-free.
2. G is almost free if and only if G is almost countably free.
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Theorem

Let G be a non-discrete Hausdorff topological group and $\kappa > 0$ a cardinal. If G contains a dense free subgroup of rank κ, then G is almost κ-free. Moreover, if $2 \leq \kappa \leq \aleph_0$, then G is almost countably free.

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2. The statement fails for non-Hausdorff groups. (Let F be a discrete free group of rank $\kappa > 0$, and let $H \neq \{1\}$ be an indiscrete group which contains no nontrivial free subgroups. Then $F \times H$ is a non-discrete non-Hausdorff group, having $F \times \{1\}$ as a dense free subgroup, which is not almost κ-free.)

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3. Not all almost κ-free groups have dense free subgroups. (For $\kappa > 0$, let G be an almost κ-free group, and let A be a discrete abelian group of cardinality $> |G|$. Then, $G \times A$ is almost κ-free but has no dense free subgroups.)

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Given a free topological group F of rank $\kappa > 0$, the following are equivalent.

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Moreover, in the above situation, if $2 \leq \kappa \leq \aleph_0$, then F is almost countably free.

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Lemma

Let $\kappa > 0$ be a cardinal, and let G be a topological group containing a dense subgroup H which is almost κ-free with respect to the induced topology. Then G is itself almost κ-free.

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Every dense subgroup of a connected semi-simple real Lie group G contains a free subgroup of rank 2 that is dense (in G).

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Theorem (Melles/Shelah)

Let T be a stable theory and M a saturated model of T, such that $|M| > |T|$. Then $\mathrm{Aut}(M)$ has a dense free subgroup of rank $2^{|M|}$.

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In particular, $\mathrm{Sym}(\Omega)$ is almost $2^{|\Omega|}$-free for any infinite set Ω.

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