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Products of elementary and idempotent matrices over integral domains ☆

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ABSTRACT

A ring R such that invertible matrices over R are products of elementary matrices, is called (after Cohn) generalized Euclidean. Extending results proved by Ruitenburg for Bézout domains, characterizations of generalized Euclidean commutative domains are obtained, that relate them with the property that singular matrices are products of idempotent matrices. This latter property is investigated, focusing on 2×2 matrices, which is not restrictive in the context of Bézout domains. It is proved that domains R, that satisfy a suitable property of ideals called (princ), are necessarily Bézout domains if 2×2 singular matrices over R are products of idempotent matrices. The class of rings satisfying (princ) includes factorial and projective-free domains. The connection with the existence of a weak Euclidean algorithm, a notion introduced by O'Meara for Dedekind domains, is also investigated.

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1. Introduction

The two following major problems concerning factorizations of square matrices over rings have been considered since the middle of the 1960's.

(P1) Characterize the integral domains R such that every square invertible matrix over R is a product of elementary matrices.
(P2) Characterize the integral domains R such that every square singular matrix over R is a product of idempotent matrices.

When R is a field, Gauss Elimination produces a factorization into elementary matrices of any invertible matrix and the structure of the general linear groups $GL_n(R)$ has been studied fairly extensively since a long while (see [11]). The investigation of integral (even non-commutative) domains satisfying the property in (P1) started in 1966 with the fundamental paper by Cohn [7], who called these domains *generalized Euclidean* (GE-rings, for short), due to the fact that Euclidean domains provide the first well-known example of GE-rings different from fields. Cohn's paper gave rise to ample and deep investigations on the structure of the general linear groups and the special linear groups over various rings.

In the same year, 1966, Howie [15] produced the starting point to attack the second problem, proving the purely set-theoretical result that every transformation of a finite set which is not a permutation can be written as a product of idempotents. One year later, J.A. Erdos [12] proved that every singular matrix over a field is a product of idempotent matrices, thus initiating the researches on problem (P2).

In 1991 John Fountain, extending in [13] Erdos' result and results by Dawlings [10] for linear transformations of vector spaces, proved the following

Theorem 1.1. *(See Fountain [13].) Let R be a principal ideal domain and $n \geqslant 2$ an integer. The following conditions are equivalent:*

($\mathrm{ID_n}$) *Every singular $n \times n$ matrix of rank $n-1$ with entries in R is a product of idempotent matrices, of rank $n-1$.*
($\mathrm{H_n}$) *For every endomorphism α of R^n of rank $n-1$, there exists an endomorphism β with $Ker(\beta) = Ker(\alpha)$ and $Im(\beta) = Im(\alpha)^*$, such that β is a product of idempotent endomorphisms of rank $n-1$.*
($\mathrm{SC_n}$) *For any pure submodules A, B of the free R-module R^n, of ranks $n-1$ and 1 respectively and such that $A \cap B = 0$, there is a sequence of direct decompositions of R^n, with $A = A_1$ and $B = B_k$:*

$$R^n = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k.$$

In condition ($\mathrm{H_n}$), which is slightly rephrased, but equivalent, to the original condition in [13], $Im(\alpha)^*$ denotes the pure closure of the submodule $Im(\alpha)$ in R^n. Fountain's proof makes use of the semigroup structure of $M_n(R)$ and of the fact that, if R is a PID, then $M_n(R)$ is an abundant semigroup. The characterization in the above theorem is used by Fountain to show that singular matrices, with entries either in a discrete valuation domain or in the ring of integers, are products of idempotent matrices.

Two years later, in 1993, Theorem 1.1 was extended by Ruitenburg in [24] to Bézout domains, that is, domains whose finitely generated ideals are principal. He called the free module R^n over the Bézout domain R *weakly complementary* if condition $(\mathrm{SC_n})$ holds. But the most relevant contribution of Ruitenburg's paper was to establish a connection of the three conditions in Fountain's theorem with the following condition $(\mathrm{GE_n})$, thus showing an intimate relationship of the two problems (P1) and (P2), when the ground ring is a Bézout domain.

$(\mathrm{GE_n})$ *Every invertible $n \times n$ matrix M is a product of elementary matrices.*

The main result in Ruitenburg's paper is the following.

Theorem 1.2. *(See Ruitenburg [24].) For a Bézout domain R the following conditions are equivalent:*

(i) for any assigned integer $n \geqslant 2$, $(\mathrm{ID_m})$ holds for every $m \leqslant n$;
(ii) for any assigned integer $n \geqslant 2$, $(\mathrm{H_m})$ holds for every $m \leqslant n$;
(iii) for any assigned integer $n \geqslant 2$, $(\mathrm{SC_m})$ holds for every $m \leqslant n$;
(iv) $(\mathrm{GE_n})$ holds for every integer $n > 0$.

The equivalence of (i), (ii) and (iii) in the above theorem was proved by Ruitenburg following the methods used by Fountain in the case of R a PID. The proof that (iii) implies (iv) is made only for $(\mathrm{GE_2})$, and then a celebrated result by Kaplansky [17, Theorem 7.1], valid for matrices over Hermite rings, allows to lift the result to $(\mathrm{GE_n})$ for an arbitrary n. That proof, as well as the proof that (iv) implies (iii), makes use of matrix theoretical methods.

Examples of generalized Euclidean domains include, besides the Euclidean ones, the domains with stable range one, hence, in particular, all semi-local domains (that is, domains with finite maximal spectrum; see [20] and [14, V.8.2]). By Ruitenburg's Theorem 1.2, the Bézout domains of this type also provide examples of rings satisfying condition $(\mathrm{ID_n})$ for all n. Cohn proved that the rings of algebraic integers of imaginary quadratic number fields that are not Euclidean are not generalized Euclidean, as well. So, among the rings of algebraic integers of $\mathbb{Q}(\sqrt{d})$ with $d < 0$, those with $d = -19, -43, -67, -163$ are examples of PID's which fail to be generalized Euclidean (see [23]).

The aim of the present paper is to give some answers to the following natural questions.

1) What can be said on problem (P1) in general or, more precisely, on the four conditions ($\mathrm{H_n}$), ($\mathrm{SC_n}$), ($\mathrm{GE_n}$), ($\mathrm{ID_n}$) and their mutual relationships, in the context of general integral domains?
2) Can we better understand generalized Euclidean domains, even in case they are Bézout domains or PID's?
3) What can be said on problem (P2) in general?

We will not investigate the problem of finding the minimum number of factors in the decomposition of a matrix as a product of invertible matrices or as a product of idempotent matrices. These kinds of questions have been discussed, for instance, by Carter and Keller [4], Laffey [19], Vaserstein and Wheland [25].

In the very recent paper [1], Alahmadi, Jain and Leroy studied the products of idempotent matrices; their main focus was on matrices over non-commutative rings, in particular, non-commutative Bézout domains.

We wish to thank the referee for having informed us about the preceding papers, and for other useful comments and suggestions.

In the preliminary Section 2, notation, terminology, basic facts and some easy results used in the paper are collected. In particular, we point out that a result proved by Laffey in [18] has an immediate generalization from Euclidean to Bézout domains; hence, for matrices over these domains, it is possible to lift condition ($\mathrm{ID_2}$) to condition ($\mathrm{ID_n}$), for all $n > 0$.

In Section 3 we define two conditions, denoted by ($\mathrm{SFC_n}$) and ($\mathrm{HF_n}$), that are suitable modifications for general domains of conditions ($\mathrm{SC_n}$) and ($\mathrm{H_n}$) above. Then we generalize Ruitenburg's result showing that, over any domain R, it is equivalent to say that the conditions ($\mathrm{GE_n}$), ($\mathrm{SFC_n}$) and ($\mathrm{HF_n}$) hold for all $n > 0$. Thus condition ($\mathrm{SC_n}$), introduced by Fountain to deal with products of idempotent matrices over PIDs, is in fact applicable to general domains, in a slightly modified version, in connection with products of elementary matrices. However, since over Bézout domains ($\mathrm{SFC_n}$) and ($\mathrm{HF_n}$) are equivalent to ($\mathrm{SC_n}$) and ($\mathrm{H_n}$), respectively, we also get the main achievement of Ruitenburg's theorem: namely, conditions ($\mathrm{GE_n}$) and ($\mathrm{ID_n}$) are equivalent over Bézout domains.

It remains to investigate condition ($\mathrm{ID_n}$) outside the class of Bézout domains. Since the case of 2×2 matrices is highly illuminating for the general situation, and in view of the generalization of Laffey's result to Bézout domains recalled in Section 2, in the remaining sections we investigate condition ($\mathrm{ID_2}$). We recall that this property was recently examined by Bhaskara Rao in [3].

Actually, the results in [18,24,3] suggest that a domain R satisfying property ($\mathrm{ID_2}$) should necessarily be a Bézout domain. In Section 4 we stress the likelihood that this feeling is correct, by examining a suitable property of a domain R, called (princ). We prove that factorial domains and projective-free domains (in particular, local domains) satisfy this property. Our main result is that a domain satisfying property (princ) and condition ($\mathrm{ID_2}$) is necessarily a Bézout domain. Note that in the case of projective-free domains, a similar result was established in [3]. As a consequence, we derive that all local domains which fail to be valuation domains provide examples of generalized Euclidean domains not satisfying condition (ID_2).

In Section 5 we characterize 2×2 matrices which are products of *basic* idempotents, that is, idempotent matrices with a zero on the diagonal. We prove that all 2×2 singular matrices over the domain R are products of basic idempotents exactly if R is a valuation domain.

Finally, in Section 6 we consider integral domains admitting a *weak Euclidean algorithm*, which O'Meara in [22] called domains satisfying the *Euclidean chain condition*; they are necessarily Bézout domains. We remark that in [22] O'Meara proved that the PIDs admitting a weak Euclidean algorithm are exactly those satisfying condition (GE_2); his proof extends *verbatim* to the case of Bézout domains. Here we give a direct proof that a Bézout domain admitting a weak Euclidean algorithm satisfies condition (ID_2). We show that some classes of domains admit a weak Euclidean algorithm, namely, special intersections of infinitely many valuation domains, and pull-back rings of the form $D + XQ[\![X]\!]$, where D is an integral domain admitting a weak Euclidean algorithm, and Q is the field of quotients of D. Note that these pull-back rings are reminiscent of the examples obtained by Chen and Leu [5], where formal power series are replaced by polynomials. As a by-product of O'Meara's result, we can show that a Bézout domain that is generalized Euclidean is also an elementary divisor ring (see [17]); the converse is not true.

2. Notation and preliminary facts

By R we will always denote an integral domain, and by $U(R)$ the multiplicative group of its invertible elements. For a domain R and an integer $n \geqslant 2$, we denote by R^n the free R-module of the column vectors $\mathbf{v} = [v_1 v_2 \dots v_n]^T$ with n coordinates $v_i \in R$. $M_n(R)$ denotes, as usual, the R-algebra of the $n \times n$ matrices with entries in R. Matrices are denoted by capital bold-face letters, like $\mathbf{M}$. Differently from Fountain and Ruitenburg, we prefer to operate with column block decompositions of matrices, so $\mathbf{M} \in M_n(R)$ will be written by columns $\mathbf{M} = [\mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_n]$. If $\mathbf{v}$ is a column vector, its transpose row vector is denoted by $\mathbf{v}^T$; $\mathbf{I}_n$ denotes the identity matrix of order n. The (column) vectors of the canonical basis of R^n are denoted by $\mathbf{e}_i$ $(1 \leqslant i \leqslant n)$. A matrix $\mathbf{M}$ is said to be *singular* if $\det(\mathbf{M}) = 0$; this amounts to say that its columns are linearly dependent over R.

The elementary $n \times n$ matrices, usually denoted by $\mathbf{E}$, are of three different types: (i) transpositions $\mathbf{P}_{ij}$ $(i \neq j)$; (ii) dilatations $\mathbf{D}_i(u)$, where $u \in U(R)$; (iii) transvections $\mathbf{T}_{ij}(r)$ $(i \neq j)$, where $r \in R$. All these matrices are invertible, since their determinants are units of R.

Recall that a Bézout domain is an integral domain whose finitely generated ideals are principal. The ring $\mathbb{Z} + X\mathbb{Q}[X]$, consisting of the polynomials over $\mathbb{Q}$ with constant term in $\mathbb{Z}$, is the standard example of a non-local Bézout domain that is not a principal ideal domain (the ideal $X\mathbb{Q}[X]$ is not finitely generated).

For modules over Bézout domains and more generally over Prüfer domains, the concept of purity (in Cohn sense) coincides with the generally weaker notion of relative divisibility, where M is a relatively divisible R-submodule of N (RD-submodule, for short) if $M \cap rN = rM$ for every $r \in R$ (see [14]). Moreover, pure submodules (in Cohn sense) of free modules of finite rank over integral domains are direct summands (see [14, VI.9.6]).

Now we prove some easy results on 2×2 singular matrices. The next proposition is folklore; its proof follows from a direct matrix computation.

Proposition 2.1. *Let R be an integral domain, $\mathbf{T} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ a non-zero and non-identity 2×2 matrix with entries in R. Then $\mathbf{T}$ is idempotent if and only if $d = 1 - a$ and $a(1-a) = bc$.*

A 2×2 matrix $\mathbf{T}$ over R is called *column–row* if there exist $a, b, x, y \in R$ such that

$$\mathbf{T} = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} xa & xb \\ ya & yb \end{pmatrix}.$$

Proposition 2.2. *Let R be an integral domain, $\mathbf{T}$ a singular matrix in $M_2(R)$, such that the ideal of R generated by the entries of its first row is principal. Then $\mathbf{T}$ is a column–row matrix. Moreover, if $\mathbf{T}$ is an idempotent column–row matrix, then also the converse is true.*

Proof. Say $\mathbf{T} = \left(\begin{smallmatrix} a_0 & b_0 \\ c & d \end{smallmatrix}\right)$, where $\langle a_0, b_0 \rangle = xR$. Then $a_0 = xa$ and $b_0 = xb$, with $\langle a, b \rangle = R$, say $1 = \lambda a + \mu b$ for suitable $\lambda, \mu \in R$. From $a_0 d - b_0 c = 0$ we get $ad = bc$. It follows that $c = c\lambda a + c\mu b = c\lambda a + \mu da$, hence we get $y = c/a \in R$. We readily conclude that $c = ya$ and $d = yb$, so $\mathbf{T} = \left(\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}\right)$ is a column–row matrix.

Assume now that $\mathbf{T}$ is idempotent and that we may write $\mathbf{T} = \left(\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}\right)$, for suitable $x, y, a, b \in R$. From $\mathbf{T}$ idempotent it follows that $yb = 1 - xa$, hence $\langle a, b \rangle = R$, so that $\langle xa, xb \rangle = xR$. □

The following lemma, whose proof is trivial, will be useful for our discussion.

Lemma 2.3. *Let R be an integral domain, $\mathbf{S} = \left(\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}\right)$ a column–row matrix ($a, b, x, y \in R$). Pick any $\mathbf{U} = \left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$ in $M_2(R)$. Then $\mathbf{US} = \left(\begin{smallmatrix} x'a & x'b \\ y'a & y'b \end{smallmatrix}\right)$, for suitable $x', y' \in R$, so it is a column–row matrix, and, if $\langle a, b \rangle$ is a principal ideal of R, then the ideal generated by the entries of the first row of $\mathbf{US}$ is principal, as well.*

Proof. By a direct computation we get

$$\mathbf{US} = \begin{pmatrix} (rx + sy)a & (rx + sy)b \\ (tx + uy)a & (tx + uy)b \end{pmatrix}. \qquad \square$$

In [18], Laffey proved that a Euclidean domain satisfies ($\mathrm{ID_n}$) for every $n > 0$. A crucial part of his proof was a reduction from ($\mathrm{ID_n}$) to ($\mathrm{ID_2}$). Bhaskara Rao, [3], observed that Laffey's reduction argument extends to the case when R is a PID. As a matter of fact, the result that follows is valid even assuming that R is a Bézout domain. We thought it convenient to point out why one can make this more general assumption.

Proposition 2.4. *Let R be a Bézout domain. If every 2×2 singular matrix with entries in R is a product of idempotent matrices, then every $n \times n$ singular matrix with entries in R is a product of idempotent matrices, for any positive integer n.*

Proof. We assume that property ($\mathrm{ID_2}$) holds and prove ($\mathrm{ID_n}$), by induction on $n \geqslant 2$. Let $\mathbf{A}$ be any singular matrix in $M_n(R)$. The starting point of Laffey's argument is to consider a row vector $\mathbf{v}^T = (v_1 \ldots v_n)$, $v_i \in R$, such that $\mathbf{v}^T\mathbf{A} = 0$. Since R is a Bézout domain, we may assume that $1 \in \langle v_1, \ldots, v_n \rangle$, i.e., $\mathbf{v}^T$ is unimodular. Then the unimodular row lemma [21, Theorem II.1], valid for Bézout domains, allows us to find a unimodular matrix $\mathbf{T}$ whose last row is $\mathbf{v}^T$. From this fact, using matrix computation we derive that every idempotent matrix $\mathbf{Y} \in M_k(R)$ is similar to a diagonal matrix $\left(\begin{smallmatrix} \mathbf{I}_h & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where h is the rank of $\mathbf{Y}$ (Lemma 1 in [18]). Then the inductive hypothesis and a matrix theoretical argument (valid over any integral domain, see [18]), shows that $\mathbf{A}$ is necessarily a product of idempotents, as soon as this property holds for every singular 2×2 matrix over R. □

To make this paper self-contained, we prove that property ($\mathrm{H_2}$) is equivalent to ($\mathrm{ID_2}$), when the ground ring is a Bézout domain. Under such circumstances, the preceding proposition ensures that also ($\mathrm{ID_n}$) holds, for every $n > 0$. Of course, this equivalence is part of Ruitenburg's Theorem 1.2.

Lemma 2.5. *Let R be an integral domain, $a, b, x, y \in R$. The matrix $\mathbf{A} = \left(\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}\right)$ is a product of idempotents whenever the matrices $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x & 0 \\ y & 0 \end{smallmatrix}\right)$ are products of idempotents.*

Proof. Just observe that

$$\begin{pmatrix} xa & xb \\ ya & yb \end{pmatrix} = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}. \qquad \square$$

Proposition 2.6. *For a Bézout domain R, property* ($\mathrm{H_2}$) *is equivalent to property* ($\mathrm{ID_2}$)*. In this case, also* ($\mathrm{ID_n}$) *is satisfied, for every $n > 0$.*

Proof. One implication is trivial. So it is enough to assume that condition ($\mathrm{H_2}$) holds, and show that any non-zero singular matrix $\mathbf{A} \in M_2(R)$ is a product of idempotent matrices. Since R is a Bézout domain, from Proposition 2.2 we get $\mathbf{A} = \left(\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}\right)$, for suitable $a, b, x, y \in R$. It suffices to show that any matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ is a product of idempotents, since, by transposition, the same will be true for $\left(\begin{smallmatrix} x & 0 \\ y & 0 \end{smallmatrix}\right)$, and for $\mathbf{A}$, by Lemma 2.5. In conclusion, we may assume that

$$\mathbf{A} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad a, b \in R.$$

Let α be the endomorphism of R^2 whose associated matrix with respect to the canonical basis is $\mathbf{A}$. By hypothesis, there exists an endomorphism β with $Ker(\beta) = Ker(\alpha)$ and $Im(\beta) = Im(\alpha)^*$, such that β is a product of idempotent endomorphisms, that is, the matrix $\mathbf{B}$ associated to β with respect to the canonical bases is a product of idempotent matrices. We claim that $\alpha = t\beta$, where t is a non-zero element of R. For, if $R^2 = \mathbf{x}R \oplus \mathbf{y}R$, where $\mathbf{y}R = Ker(\alpha)$, then $\alpha(\mathbf{x}) = t\beta(\mathbf{x})$ for a suitable non-zero $at \in R$. Thus, given $\mathbf{v} = r\mathbf{x} + s\mathbf{y} \in R^2$ $(r, s \in R)$, we get

$$\alpha(\mathbf{v}) = r\alpha(\mathbf{x}) + s\alpha(\mathbf{y}) = ra\beta(\mathbf{x}) = t\big(r\beta(\mathbf{x}) + s\beta(\mathbf{y})\big) = t\beta(\mathbf{v}).$$

Now, from a direct computation, we see that $\mathbf{A} = t\mathbf{B}$ equals $\mathbf{UB}$, where

$$\mathbf{U} = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t-1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

and the factors on the right are idempotent matrices over any domain, by Proposition 2.1. It follows that $\mathbf{A}$ is a product of idempotent matrices. The last statement follows from Proposition 2.4. □

3. A generalization of Ruitenburg's result

In the next Theorem 3.4 we will deal with two equivalent properties, denoted by ($\mathrm{SFC_n}$) and ($\mathrm{HF_n}$), that generalize the above defined properties ($\mathrm{SC_n}$) and ($\mathrm{H_n}$), respectively. We immediately remark that these properties coincide in the context of Bézout domains.

Theorem 3.4 generalizes Ruitenburg's result, valid for Bézout domains, to any integral domain; we give a direct and self-contained proof, that doesn't use semigroup theoretical methods.

We need three preliminary lemmas.

Lemma 3.1. *Let $\mathbf{E} = [\mathbf{A}_1\mathbf{A}_2]$ and $\mathbf{F} = [\mathbf{B}_1\mathbf{B}_2]$ be two invertible $n \times n$ matrices, with entries in a domain R, and $\mathbf{A}_1$ and $\mathbf{B}_1$ blocks of size $n \times r$.*

(1) If $\mathbf{A}_2 = \mathbf{B}_2$, the last $n - r$ columns of the matrix $\mathbf{F}^{-1}\mathbf{E}$ are equal to the vectors $\mathbf{e}_{r+1}, \dots, \mathbf{e}_n$.
(2) If $\mathbf{A}_1 = \mathbf{B}_1$, the first r columns of the matrix $\mathbf{E}^{-1}\mathbf{F}$ are equal to the vectors $\mathbf{e}_1, \dots, \mathbf{e}_r$.

Proof. (1) Let $\mathbf{F}^{-1} = [\,\mathbf{X}_1^T \quad \mathbf{X}_2^T\,]^T$; then from $\mathbf{F}^{-1}\mathbf{F} = \mathbf{I}_n$ we get $\mathbf{X}_1\mathbf{B}_2 = \mathbf{O}$ and $\mathbf{X}_2\mathbf{B}_2 = \mathbf{I}_{n-r}$. From these equalities and from $\mathbf{A}_2 = \mathbf{B}_2$ it follows immediately that the last $n - r$ columns of $\mathbf{F}^{-1}\mathbf{E}$ are equal to the coordinate vectors $\mathbf{e}_{r+1}, \dots, \mathbf{e}_n$.

(2) Let $\mathbf{E}^{-1} = [\,\mathbf{Y}_1^T \quad \mathbf{Y}_2^T\,]^T$; then $\mathbf{E}^{-1}\mathbf{E} = \mathbf{I}_n$ implies that $\mathbf{Y}_1\mathbf{A}_1 = \mathbf{I}_r$ and $\mathbf{Y}_2\mathbf{A}_1 = \mathbf{O}$. From this equality and from $\mathbf{A}_1 = \mathbf{B}_1$ it follows that the first r columns of $\mathbf{E}^{-1}\mathbf{F}$ equal $\mathbf{e}_1, \dots, \mathbf{e}_r$. □

Lemma 3.2. *Let R be a domain and $R^n = X \oplus Y$ a free direct decomposition, where $X = \bigoplus_{1 \leqslant i \leqslant r} \mathbf{a}_i R$ and $Y = \bigoplus_{r+1 \leqslant i \leqslant n} \mathbf{a}_i R$. Let $\mathbf{M} = [\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n]$ be the associated invertible matrix. If $\mathbf{E}$ is an elementary matrix, then the direct decomposition of R^n associated to the product matrix $\mathbf{ME}$ is:*

(a) $R^n = X \oplus Y$ if $\mathbf{E}$ is either a dilatation $\mathbf{D}_j(u)$ or a transvection $\mathbf{T}_{ij}(r)$ with $i, j \leqslant r$ or $i, j > r$;
(b) $R^n = X \oplus Y'$ if $\mathbf{E}$ is a transvection $\mathbf{T}_{ij}(r)$ with $i \leqslant r < j$;
(c) $R^n = X' \oplus Y$ if $\mathbf{E}$ is a transvection $\mathbf{T}_{ij}(r)$ with $j \leqslant r < i$.

Proof. It is enough to note that in case (a) the post-multiplication by $\mathbf{E}$ acts as a modification of the j-th column, that does not change the summands X and Y. In case (b), the modification of the j-th column of $\mathbf{M}$ modifies the summand Y into a new summand Y', but the summand X remains the same, and in case (c) the reverse happens. □

The next result follows from the general theory of semigroups (see Lemma 7 in [24], which refers to Theorem 2.17 in [6]); we give a direct proof in our particular setting.

Lemma 3.3. *Let R be an integral domain, and α, β two endomorphisms of R^n such that $Im(\alpha)$ and $Im(\beta)$ are free summands of R^n of the same rank. If the endomorphism $\beta\alpha$ satisfies $Ker(\beta\alpha) = Ker(\alpha)$ and $Im(\beta\alpha) = Im(\beta)$, then there exists an idempotent endomorphism η such that $Im(\eta) = Im(\alpha)$ and $Ker(\eta) = Ker(\beta)$.*

Proof. We firstly show that $Im(\alpha) \cap Ker(\beta) = 0$. For, let $z = \alpha(x)$ be such that $\beta(z) = \beta(\alpha(x)) = 0$; then $x \in Ker(\beta\alpha) = Ker(\alpha)$, so $0 = \alpha(x) = z$. So we have that $Im(\alpha) \oplus Ker(\beta) \leqslant R^n$. But $Im(\alpha)$ is a summand of R^n, by hypothesis, and $Ker(\beta)$ is a summand by the projectivity of $Im(\beta\alpha) = Im(\beta)$. We claim that from these facts the equality $Im(\alpha) \oplus Ker(\beta) = R^n$ follows.

In fact, given $0 \neq z \in R^n$ and since $rk(Im(\alpha) \oplus Ker(\beta)) = n$, there exists $0 \neq r \in R$ such that $rz = \alpha(x) + y$, with $\beta(y) = 0$. Then $r\beta(z) = \beta(rz) = \beta\alpha(x)$, and since $\beta(z) = \beta\alpha(x')$ for some $x' \in R^n$, we get that $x - rx' \in Ker(\beta\alpha) = Ker(\alpha)$. So $r\alpha(x') = \alpha(rx') = \alpha(x)$, and consequently the RD-divisibility of $Ker(\beta)$ in R^n gives that $y = ry'$ for some $y' \in Ker(\beta)$. The conclusion is that $z = \alpha(x') + y' \in Im(\alpha) \oplus Ker(\beta)$, as claimed. Therefore the required idempotent η is the projection onto $Im(\alpha)$ with kernel $Ker(\beta)$. □

Theorem 3.4. *For an integral domain R the following conditions are equivalent:*

($\mathrm{HF_n}$) *For any free direct summands A, B of the free R-module R^n, of ranks r and $n-r$ respectively ($1 \leqslant r < n$), there exists an endomorphism β of R^n with $Ker(\beta) = B$ and $Im(\beta) = A$, which is a product of idempotent endomorphisms of rank r.*

($\mathrm{SFC_n}$) *For any free direct summands A, B of the free R-module R^n, of ranks r and $n-r$ respectively ($1 \leqslant r < n$), there exist direct decompositions of R^n, with $A = A_1$ and $B = B_k$: $R^n = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k$.*

($\mathrm{GE_n}$) *Every invertible $n \times n$ matrix M is a product of elementary matrices.*

Proof. ($\mathrm{HF_n}$) $\Rightarrow$ ($\mathrm{SFC_n}$). Let A, B be free direct summands of R^n of ranks r and $n-r$, respectively. By hypothesis, there exists an endomorphism β of R^n with $Ker(\beta) = B$ and $Im(\beta) = A$, which is a product of idempotent endomorphisms of the same rank r, say $\beta = \pi_1 \dots \pi_k$. For each $1 \leqslant i \leqslant k$, let $R^n = A_i \oplus B_i$ be the direct decomposition induced by the idempotent endomorphism π_i, that is, $A_i = Im(\pi_i)$ and $B_i = Ker(\pi_i)$. Since all the ranks equal r, we get $A = A_1$ and $B = B_k$, and also $A_i = Im(\pi_i \cdot \pi_{i+1})$ and $B_{i+1} = Ker(\pi_i \cdot \pi_{i+1})$, for every $i < k$. By Lemma 3.3, there exists an idempotent endomorphism η_i of R^n such that $Ker(\eta_i) = B_i$ and $Im(\eta_i) = A_{i+1}$, hence $R^n = A_{i+1} \oplus B_i$, as desired.

($\mathrm{SFC_n}$) $\Rightarrow$ ($\mathrm{HF_n}$). Let A, B be free direct summands of R^n of ranks r and $n-r$, respectively. By hypothesis, there is a sequence of direct decompositions of R^n, with $A = A_1$ and $B = B_k$:

$$R^n = A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k.$$

For each $i \leqslant k$, let $\pi_i : R^n \to R^n$ be the projection such that $Im(\pi_i) = A_i$ and $Ker(\pi_i) = B_i$. The map $\beta = \pi_1 \dots \pi_k$ is a product of idempotents, with $Ker(\beta) \subseteq B = Ker(\pi_k)$ and $Im(\beta) \subseteq A = Im(\pi_1)$. We must show that these two inclusions are actually equalities. We induct on k, the case $k = 1$ being trivial. Let $k > 1$; by the inductive hypothesis, setting $\gamma = \pi_2 \dots \pi_k$, we have $Ker(\gamma) = B$ and $Im(\gamma) = A_2$ (note that A_2 is a free summand of R^2, being isomorphic to A). Since $\beta = \pi_1 \cdot \gamma$, we deduce:

$$A = \pi_1\bigl(R^2\bigr) = \pi_1(A_2 \oplus B_1) = \pi_1(A_2) = \pi_1\bigl(\gamma\bigl(R^2\bigr)\bigr) = \beta\bigl(R^2\bigr),$$

and therefore $Im(\beta) = A$. If $Ker(\beta)$ strictly contains B, then $Im(\beta)$ has rank strictly less than r, contradicting the fact that $Im(\beta) = A$. We conclude that $Ker(\beta) = B$.

($\mathrm{SFC_n}$) $\Rightarrow$ ($\mathrm{GE_n}$). We want to show that an invertible matrix $\mathbf{M} = [\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n] \in M_n(R)$ ($n \geqslant 1$) is a product of elementary matrices. We make induction on n, the case $n = 1$ being trivial. So assume $n \geqslant 2$. Without loss of generality, we will assume that $(\bigoplus_{1 \leqslant i \leqslant n-1} \mathbf{a}_i R) \cap \mathbf{e}_n R = 0$, where $\mathbf{e}_n$ is the last coordinate vector: for, if $\mathbf{e}_n = \sum_{1 \leqslant i \leqslant n} r_i \mathbf{a}_i$ ($r_i \in R$), choose $r_j \neq 0$; then the matrix $\mathbf{MP}_{jn}$ satisfies the required condition, and $\mathbf{M}$ is a product of elementary matrices if and only if such is $\mathbf{MP}_{jn}$. Let $A = \bigoplus_{1 \leqslant i \leqslant n-1} \mathbf{a}_i R$ and $B = \mathbf{e}_n R$; then the condition ($\mathrm{SFC_n}$) provides a sequence of direct decompositions of R^n:

$$A_1 \oplus B_1 = A_2 \oplus B_1 = A_2 \oplus B_2 = \cdots = A_k \oplus B_{k-1} = A_k \oplus B_k \tag{1}$$

where $A = A_1$ and $B = B_k$. For $1 \leqslant j \leqslant k$, let $A_j = \bigoplus_{1 \leqslant i \leqslant n-1} \mathbf{a}_{ij} R$ and $B_j = \mathbf{b}_j R$, where we set $\mathbf{a}_i = \mathbf{a}_{i1}$ for all i and $\mathbf{b}_k = \mathbf{e}_n$.

Let us consider the following matrices associated to the above direct decompositions:

$$\mathbf{E}_j = [\mathbf{a}_{1j}\mathbf{a}_{2j} \dots \mathbf{a}_{n-1j}\mathbf{b}_j], \quad 1 \leqslant j \leqslant k$$
$$\mathbf{F}_j = [\mathbf{a}_{1,j+1}\mathbf{a}_{2,j+1} \dots \mathbf{a}_{n-1,j+1}\mathbf{b}_j], \quad 1 \leqslant j \leqslant k-1$$

and set $\mathbf{F_0} = \mathbf{M}$ (i.e., $\mathbf{b}_0 = \mathbf{a}_n$). All the matrices $\mathbf{E}_j$ and $\mathbf{F}_j$ are invertible. Consider now the matrices

$$\mathbf{P}_j = \mathbf{F}_j^{-1}\mathbf{E}_j, \quad 1 \leqslant j \leqslant k$$
$$\mathbf{Q}_j = \mathbf{E}_{j+1}^{-1}\mathbf{F}_j, \quad 0 \leqslant j \leqslant k-1.$$

We claim that the matrices $\mathbf{P}_j$ and $\mathbf{Q}_j$ are products of elementary matrices. In fact, Lemma 3.1, applied with $r = n-1$, shows that the last column of each matrix $\mathbf{P}_j$ equals the vector $\mathbf{e}_n$, while the first $n-1$ columns of each matrix $\mathbf{Q}_j$ are equal to the vectors $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$. The principal submatrix of $\mathbf{P}_j$ obtained by deleting last row and column is an invertible matrix of size $n-1$, so by induction it is a product of elementary matrices. Then standard arguments using Gauss elimination show that $\mathbf{P}_j$ itself is a product of elementary matrices. The last row of each matrix $\mathbf{Q}_j$ is a vector $u_j\mathbf{e}_n^T$, with u_j necessarily a unit, so a similar argument applies to prove that the matrices $\mathbf{Q}_j$ are also product of elementary matrices.

Now, starting from the equality $\mathbf{M} = \mathbf{F}_0 = \mathbf{E}_1\mathbf{Q}_0$, we obtain the factorizations:

$$\begin{aligned}\mathbf{M} &= (\mathbf{E}_k\mathbf{E}_k^{-1})(\mathbf{F}_{k-1}\mathbf{F}_{k-1}^{-1}) \dots (\mathbf{E}_2\mathbf{E}_2^{-1})(\mathbf{F}_1\mathbf{F}_1^{-1})\mathbf{E}_1\mathbf{Q}_0 \\ &= \mathbf{E}_k(\mathbf{E}_k^{-1}\mathbf{F}_{k-1})(\mathbf{F}_{k-1}^{-1}\mathbf{E}_{k-1}) \dots (\mathbf{E}_2^{-1}\mathbf{F}_1)(\mathbf{F}_1^{-1}\mathbf{E}_1)\mathbf{Q}_0 \\ &= \mathbf{E}_k\mathbf{Q}_{k-1}\mathbf{P}_{k-1} \dots \mathbf{Q}_1\mathbf{P}_1\mathbf{Q}_0.\end{aligned}$$

Since the last column of $\mathbf{E}_k$ is the vector $\mathbf{e}_n$, the inductive argument used above shows that it is a product of elementary matrices, hence the same is true for the matrix $\mathbf{M}$.

($\mathrm{GE_n}$) $\Rightarrow$ ($\mathrm{SFC_n}$). Let A, B be free direct summands of R^n, of ranks r and $n-r$ respectively. Let $R^n = A \oplus X = Y \oplus B$ and let $\mathbf{M}$ and $\mathbf{N}$ be the matrices associated to the direct decompositions $R^n = A \oplus X$ and $R^n = Y \oplus B$, respectively. The invertible matrix $\mathbf{X} = \mathbf{N}^{-1}\mathbf{M}$ is a product of elementary matrices by hypothesis, so

$\mathbf{M} = \mathbf{N}\mathbf{X} = \mathbf{N}\mathbf{E}_1\mathbf{E}_2 \ldots \mathbf{E}_k$, where we can assume each matrix $\mathbf{E}_i$ being either a dilatation or a transvection, since it is well-known that any permutation matrix is a product of matrices of this sort. Lemma 3.2 shows that each post-multiplication of $\mathbf{N}\mathbf{E}_1\mathbf{E}_2 \ldots \mathbf{E}_{i-1}$ by $\mathbf{E}_i$ gives rise to an invertible matrix associated to a direct decomposition of R^n that modifies, in the previous decomposition, either the summand of rank r, or the summand of rank $n-r$. So we pass from the decomposition $R^n = Y \oplus B$ associated to $\mathbf{N}$ to the decomposition $R^n = A \oplus X$ associated to $\mathbf{M}$ with the desired sequence of free decompositions. □

Note that some arguments in Theorem 3.4 have been inspired by Ruitenburg's techniques. Also note that, proving ($\mathrm{SFC_n}$) ⇒ ($\mathrm{GE_n}$), we use only part of the strength of condition ($\mathrm{SFC_n}$), when we assume that the summand A has rank $n-1$. Moreover, the reverse implication is proved for a summand A of arbitrary rank. As a consequence, we see that, if condition ($\mathrm{SFC_n}$) is assumed just for $rk(A) = n-1$, then it holds for A of arbitrary rank.

When R is a Bézout domain, it is readily verified that the above properties ($\mathrm{SFC_n}$) and ($\mathrm{HF_n}$) coincide, respectively, with ($\mathrm{SC_n}$) and ($\mathrm{H_n}$), defined in the introduction. This fact follows since, as recalled in the preceding section, the concepts of purity and relative divisibility coincide for modules over Prüfer domains, and pure submodules of free modules of finite rank over integral domains are direct summands.

As a consequence of the preceding theorem, we get the main achievement of Ruitenburg's Theorem 1.2.

Theorem 3.5 *(Ruitenburg). Let R be a Bézout domain. Then R is generalized Euclidean if and only if every singular matrix with entries in R is a product of idempotent matrices.*

Proof. Assume that R satisfies ($\mathrm{GE_2}$); then, by Theorem 3.4, it satisfies ($\mathrm{HF_2}$), which, over Bézout domains, is equivalent to ($\mathrm{H_2}$). Thus ($\mathrm{ID_2}$) holds, by Proposition 2.6, and therefore, from Proposition 2.4 we get ($\mathrm{ID_n}$), for all $n > 0$.

Conversely, from ($\mathrm{ID_2}$) we get ($\mathrm{HF_2}$), whence Theorem 3.4 yields ($\mathrm{GE_2}$). Then, in view of the above recalled Kaplansky's result in [17], ($\mathrm{GE_n}$) is valid for all $n > 0$. □

4. The properties (princ) and ($\mathrm{ID_2}$)

In the remainder of the paper we will deal with products of idempotent matrices, a notion related to products of elementary matrices, via Ruitenburg's Theorem 1.2, or its more general form Theorem 3.4. Actually, motivated by Proposition 2.4, we will focus on 2×2 idempotent matrices.

The results in [18,24,3] suggest the following natural conjecture: "If R satisfies property ($\mathrm{ID_2}$), then it is a Bézout domain."

The purpose of this section is to add likelihood to this conjecture, by an examination of the following property of an integral domain R:

(princ) *If* $a, b, c \in R$ *satisfy the relation* $a(1-a) = bc$*, then the ideals* $\langle a, b\rangle$*,* $\langle a, c\rangle$*,* $\langle a-1, b\rangle$*,* $\langle a-1, c\rangle$ *of* R *are all principal.*

Obviously, any Bézout domain satisfies the above property. Let us see that (princ) is equivalent to a weaker condition.

Proposition 4.1. *An integral domain* R *satisfies (princ) if and only if the relation* $a(1-a) = bc$ $(a, b, c \in R)$ *implies that* $\langle a, b\rangle$ *is a principal ideal of* R*.*

Proof. Only sufficiency needs a proof. To simplify the notation, we write $d = 1 - a$. Assume that $\langle a, b\rangle = xR$. We must prove that $\langle a, c\rangle$, $\langle d, b\rangle$, $\langle d, c\rangle$ are principal ideals. We have $a = xa_1$ and $b = xb_1$, with $\langle a_1, b_1\rangle = R$, say $1 = \lambda a_1 + \mu b_1$ for suitable $\lambda, \mu \in R$. From $ad - bc = 0$ we get $a_1 d = b_1 c$. It follows that $c = c\lambda a_1 + c\mu b = c\lambda a_1 + \mu d a_1$, hence we get $y = c/a_1 \in R$. We readily conclude that $c = ya_1$ and $d = yb_1$, so $\langle d, c\rangle = yR$. Moreover the above computations yield

$$\langle a, c\rangle = \langle xa_1, ya_1\rangle, \qquad \langle d, b\rangle = \langle yb_1, xb_1\rangle.$$

So, if we show that $\langle x, y\rangle = R$, we can conclude that $\langle a, c\rangle = a_1 R$, $\langle d, b\rangle = b_1 R$, as desired. And, in fact, $yb_1 = d = 1 - a = 1 - xa_1$ shows that $1 \in \langle x, y\rangle$. □

Besides Bézout domains, other large classes of integral domains satisfy (princ).

Proposition 4.2. *An integral domain* R *satisfies (princ) whenever its projective two-generated ideals are free, i.e., principal.*

Proof. Let $a(1-a) = bc$ for some $a, b, c \in R$. In order to show that R satisfies (princ), by Proposition 4.1 if suffices to verify that the ideal $\langle a, b\rangle$ is principal. We may assume that $a \neq 0$. From $a(1-a) = bc$ it follows that $aR = \langle a^2, bc\rangle$, hence

$$\langle a, b\rangle\langle a, c\rangle = \left\langle a^2, ac, ab, bc\right\rangle = aR.$$

Thus $\langle a, b\rangle$ is an invertible ideal, with inverse $\langle 1, c/a\rangle$, hence, by hypothesis, it is free. We conclude that R satisfies (princ). □

Recall that a ring R is called *projective-free* if finitely generated projective R-modules are free.

Proposition 4.3. *An integral domain* R *satisfies (princ) when*

(a) R *is factorial;*
(b) R *is projective-free.*

In particular, a local domain satisfies (princ).

Proof. (a) If R is factorial, then its invertible ideals are free. So R satisfies (princ) by Proposition 4.2.

(b) If R is projective-free, then it satisfies (princ) by Proposition 4.2.

Finally, since a local domain R is projective-free (see [14, VI.1.9]), the last claim follows from (b). However, we can readily give a direct proof. In fact, let $a(1-a) = bc$ for some a, b, c in the local domain R. If a is a unit, then $\langle a, b\rangle = R$. If not, then $1-a$ is a unit, so $a = bc(1-a)^{-1}$ yields $\langle a, b\rangle = bR$. □

Let us observe that there exist factorial domains that are not projective-free. We thank Roger Wiegand for suggesting to us the following example.

Example 4.4. Let X, Y, Z be indeterminates over the real numbers $\mathbb{R}$. Consider the ring $D = \mathbb{R}[X,Y,Z]$ and its principal ideal I generated by $X^2 + Y^2 + Z^2 - 1$. Then the coordinate ring of the real 3-dimensional sphere $R = D/I$ is factorial, but not projective-free.

The next proposition shows a close relationship between idempotent column–row matrices and property (princ).

Proposition 4.5. *Let R be an integral domain. Then every idempotent matrix $\mathbf{T} \in M_2(R)$ is a column–row matrix if and only if R satisfies (princ). In such case, the entries of rows and columns of an idempotent matrix generate principal ideals.*

Proof. Assume that R satisfies (princ), and pick any idempotent matrix, say $\mathbf{T} = \left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$. Then from $a(1-a) = bc$ and property (princ) we get that the ideals $\langle a, b\rangle$, $\langle a, c\rangle$, $\langle a-1, b\rangle$, $\langle a-1, c\rangle$ of R are all principal. Since we are in the position to apply the first part of Proposition 2.2, we conclude that $\mathbf{T}$ is a column–row matrix. Conversely, assume that every idempotent matrix $\mathbf{T} = \left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$ is column–row. Then the second part of Proposition 2.2 shows that $\langle a, b\rangle$ is a principal ideal. We readily conclude that R satisfies (princ). □

For convenience, we denote by $ID_2(R)$ the set of the 2×2 matrices with entries in R that are products of idempotent matrices. So, under this notation, an integral domain has property (ID_2) exactly when every singular matrix of $M_2(R)$ lies in $ID_2(R)$.

In the next theorem we prove that, if R satisfies both (ID_2) and (princ), then R is necessarily a Bézout domain.

Theorem 4.6. *Let R be an integral domain that satisfies property (princ). Then every $\mathbf{S} \in ID_2(R)$ is a column–row matrix and the entries of its first row generate a principal ideal. As a consequence, if R satisfies property* (ID_2)*, then R is necessarily a Bézout domain.*

Proof. If $\mathbf{S} \in ID_2(R)$, then $\mathbf{S}$ may be obtained starting with an idempotent matrix $\mathbf{T}$ and multiplying on the left by a suitable matrix (product of idempotents). Proposition 4.5 shows that $\mathbf{T} = \left(\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}\right)$, for some $a, b, x, y \in R$, where $\langle xa, xb\rangle$ is a principal ideal. Then from Lemma 2.3 we get $\mathbf{S} = \left(\begin{smallmatrix} x'a & x'b \\ y'a & y'b \end{smallmatrix}\right)$ for suitable $x', y' \in R$, and $\langle x'a, x'b\rangle$ is a principal ideal. If now R is not a Bézout domain, we pick a two-generated non-principal ideal, say $\langle f, g\rangle$. Then the matrix $\left(\begin{smallmatrix} f & g \\ 0 & 0 \end{smallmatrix}\right)$ cannot lie in $ID_2(R)$, so R does not satisfy property (ID_2). □

From the above theorem and Proposition 4.3 we immediately get the following corollary.

Corollary 4.7. *If the integral domain R is either factorial or projective-free and satisfies* (ID_2)*, then R is a Bézout domain.*

We recall that for R projective-free the above result was proved by Bhaskara Rao [3], using different techniques.

Example 4.8. We give an example of a Dedekind domain R that does not satisfy (princ). Let $R = \mathbb{Z}[\sqrt{-5}\,]$, and consider the matrix

$$\mathbf{T} = \begin{pmatrix} 3 & 1+\sqrt{-5} \\ -1+\sqrt{-5} & -2 \end{pmatrix}.$$

Then $\mathbf{T}$ is idempotent, since $3(1-3) = (1+\sqrt{-5}\,)(-1+\sqrt{-5}\,)$, and $\langle 3, 1+\sqrt{-5}\,\rangle$ is not a principal ideal of R. This example also shows that, when R is not a factorial domain, a matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$, with $\langle a, b\rangle$ non-principal, may be a product of idempotents. In fact

$$\begin{pmatrix} 3 & 1+\sqrt{-5} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1+\sqrt{-5} \\ -1+\sqrt{-5} & -2 \end{pmatrix} \in ID_2(R).$$

Example 4.9. It is worth giving an example of an integral domain that satisfies (princ) and is neither factorial nor local. The required example is $\mathbb{Z}[\sqrt{-3}\,]$. It is not integrally closed, hence it cannot satisfy (ID_2), by virtue of Theorem 4.6. Since $\mathbb{Z}[\sqrt{-3}\,]$ is close to be factorial (its integral closure is a Euclidean domain), it is somehow expectable that it should satisfy (princ). However, to show this fact is far from being straightforward. We are grateful to Umberto Zannier for sending us, in a private communication, a detailed proof that $\mathbb{Z}[\sqrt{-3}\,]$ satisfies (princ). We omit his nice argument, since it is concerned with topics extraneous to the present paper.

5. Basic idempotents

In this section we examine a natural subset $\mathcal{B}$ of $ID_2(R)$. From our discussion we get an improvement of the known fact that a valuation domain satisfies property (ID_2).

A singular matrix $\left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right) \in M_2(R)$ is called a *basic idempotent* if the entry a is either 0 or 1. Thus a basic idempotent has one of the following forms

$$\begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ t & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix}; \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix}$$

for a suitable $t \in R$.

We denote by $\mathcal{B}$ the set of 2×2 matrices over R that are products of basic idempotents.

We say that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$ has proportional rows and columns if the following conditions are both satisfied

(i) either $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) \in R\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) \in R\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$;
(ii) either $(\,a \quad b\,) \in R(\,c \quad d\,)$ or $(\,c \quad d\,) \in R(\,a \quad b\,)$.

Manifestly, every basic idempotent has proportional rows and columns. It is straightforward to verify that $\mathbf{T} \in M_2(R)$ has proportional rows and columns if and only if we may write $\mathbf{T} = \left(\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}\right)$ where either $a \in Rb$ or $b \in Ra$ and either $x \in Ry$ or $y \in Rx$. In particular, $\mathbf{T}$ is a column–row matrix.

Theorem 5.1. *Let R be an integral domain. A matrix $\mathbf{T} \in M_2(R)$ is a product of basic idempotents if and only if $\mathbf{T}$ has proportional rows and columns.*

Proof. We firstly show that a matrix $\mathbf{T} = \left(\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}\right)$, where either $a \in Rb$ or $b \in Ra$ and either $x \in Ry$ or $y \in Rx$, lies in $\mathcal{B}$. We start with the case where $b = at$ for a suitable $t \in R$. Then

$$\begin{pmatrix} xa & xta \\ ya & yta \end{pmatrix} = \begin{pmatrix} xa & 0 \\ ya & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}.$$

Hence it suffices to check that $\left(\begin{smallmatrix} xa & 0 \\ ya & 0 \end{smallmatrix}\right) \in \mathcal{B}$. And in fact either

$$\begin{pmatrix} xa & 0 \\ ya & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y/x & 0 \end{pmatrix} \begin{pmatrix} xa & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y/x & 0 \end{pmatrix} \begin{pmatrix} 1 & xa-1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

or

$$\begin{pmatrix} xa & 0 \\ ya & 0 \end{pmatrix} = \begin{pmatrix} 0 & x/y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ya & 0 \end{pmatrix},$$

according to whether $y/x \in R$ or $x/y \in R$.

We now consider the case $a = br$, where $r \in R$. Then

$$\begin{pmatrix} xrb & xb \\ yrb & yb \end{pmatrix} = \begin{pmatrix} 0 & xb \\ 0 & yb \end{pmatrix} \begin{pmatrix} 0 & 0 \\ r & 1 \end{pmatrix},$$

and $\left(\begin{smallmatrix} 0 & xb \\ 0 & yb \end{smallmatrix}\right) \in \mathcal{B}$, since either

$$\begin{pmatrix} 0 & xb \\ 0 & yb \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y/x & 0 \end{pmatrix} \begin{pmatrix} 0 & xb \\ 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & xb \\ 0 & yb \end{pmatrix} = \begin{pmatrix} 0 & x/y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ yb-1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

according to whether $y/x \in R$ or $x/y \in R$.

Conversely, let us show that any matrix in $\mathcal{B}$ has proportional rows and columns. It suffices to prove that the proportionality of rows and columns is preserved when we multiply on the left by a basic idempotent. So take any $\mathbf{T} = \left(\begin{smallmatrix} xa & xb \\ ya & yb \end{smallmatrix}\right)$, where either $a \in Rb$ or $b \in Ra$ and either $x \in Ry$ or $y \in R$. It is readily seen that the matrices

$$\begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix} \mathbf{T} = \begin{pmatrix} xa & xb \\ txa & txb \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} \mathbf{T} = \begin{pmatrix} a(x+ty) & b(x+ty) \\ 0 & 0 \end{pmatrix}$$

have proportional rows and columns. A similar computation shows that also the matrices $\left(\begin{smallmatrix} 0 & t \\ 0 & 1 \end{smallmatrix}\right)\mathbf{T}$ and $\left(\begin{smallmatrix} 0 & 0 \\ t & 1 \end{smallmatrix}\right)\mathbf{T}$ have proportional rows and columns. □

The preceding theorem has a relevant consequence.

Theorem 5.2. *Let R be an integral domain. Every singular matrix $\mathbf{T} \in M_2(R)$ is a product of basic idempotents if and only if R is a valuation domain.*

Proof. We firstly assume that R is not a valuation domain. Pick elements $a, b \in R$ such that $a \notin bR$ and $b \notin aR$. Then Theorem 5.1 shows that the matrix $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ doesn't lie in $\mathcal{B}$.

Conversely, let R be a valuation domain and take any singular 2×2 matrix $\mathbf{T} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ over R. Assume that $b = at \in aR$ and $c = ar \in aR$. Then $ad = bc$ shows that $d = tra$, and so, manifestly, $\mathbf{T} = \left(\begin{smallmatrix} a & ta \\ ra & tra \end{smallmatrix}\right)$ has proportional rows and columns. In a similar way one treats the cases where $a \in bR$, $a \in cR$, etc., in each case concluding that $\mathbf{T}$ has proportional rows and columns. Therefore $\mathbf{T} \in \mathcal{B}$, in view of Theorem 4.6. □

Corollary 5.3. *Let R be a local domain. Then R satisfies property* (ID_2) *if and only if R is a valuation domain.*

Proof. By Theorem 5.1, a valuation domain satisfies (ID_2). Moreover, a local domain R that satisfies (ID_2) must be a Bézout domain, by Theorem 4.6 and Proposition 4.3. Hence R is a valuation domain. □

Since it is well-known, and easily proved, that any local domain satisfies (GE_2), the preceding corollary shows that Theorem 3.5 is no longer valid outside the class of Bézout domains.

6. Weak Euclidean algorithm

In the paper [22], O'Meara examined the natural notion of *Euclidean chain condition* for Dedekind domains. Here we extend O'Meara's definition to arbitrary domains; note that the Dedekind condition is irrelevant. We try to use a terminology as simple as possible.

Let R be an integral domain, and pick two non-zero elements $a, b \in R$. We say that a, b satisfy a *weak Euclidean algorithm* (or weak algorithm, for short) if there is a finite sequence of relations

$$r_i = q_{i+1}r_{i+1} + r_{i+2}, \quad r_i, q_i \in R, \ -1 \leqslant i \leqslant n-2,$$

such that $b = r_{-1}$, $a = r_0$, $r_{n-1} \neq 0$ and $r_n = 0$.

We say that R admits a weak Euclidean algorithm if this phenomenon happens for any pair of non-zero elements $a, b \in R$. (In O'Meara's terminology, R satisfies the Euclidean chain condition.)

The algorithm implies that $\langle a, b\rangle = r_{n-1}R$, hence any domain admitting a weak Euclidean algorithm must be a Bézout domain.

We remark two useful, readily verified facts:

(i) $a, b \in R$ satisfy a weak algorithm if one of them divides the other;
(ii) if $a, b \in R$ satisfy a weak algorithm, then the same is true for $u_1 a$, $u_2 b$ for all units $u_1, u_2 \in R$.

It is easy to show that R admits a weak Euclidean algorithm when R is a valuation domain.

Remark 1. We remark that in the paper [8], G. Cooke generalized the notion of Euclidean domain, and applied it to the rings of integers in quadratic number fields in [9]. Several papers have followed his research, see, e.g., [5]. In Cooke's terminology (see [8]), a domain R that satisfies a weak Euclidean algorithm is an ω-stage Euclidean domain, if we consider the trivial norm N on R defined by $N(0) = 0$ and $N(a) = 1$ for $0 \neq a \in R$, hence our discussion is valid for those kinds of rings.

We recall that O'Meara proved the following nice result, that we state in our terminology.

Theorem 6.1. *(See O'Meara [22, Theorem 14.3].) A principal ideal domain R admits a weak Euclidean algorithm if and only if R satisfies property* (GE_2)*.*

Actually, we remark that the above theorem extends *verbatim* to Bézout domains R. From Theorem 3.5 it follows that such R satisfies property (ID_2), as well. We thought it convenient to prove directly this last result.

Theorem 6.2. *If an integral domain R admits a weak Euclidean algorithm, then it satisfies property* (ID_2).

Proof. Since R is automatically a Bézout domain, by Proposition 2.2 any singular 2×2 matrix $\mathbf{T}$ over R has the form $\mathbf{T} = \begin{pmatrix} xa & xb \\ ya & yb \end{pmatrix}$ for suitable $a, b, x, y \in R$. Then, using Lemma 2.5, we see that R satisfies (ID_2) as soon as every matrix of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ lies in $ID_2(R)$, so that, by transposition, also $\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \in ID_2(R)$ for all $x, y \in R$. So we take $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and prove that it lies in $ID_2(R)$. Now we apply to a, b the weak algorithm, producing the elements $r_1, \ldots, r_{n-1}, r_n = 0$. At the first step we get $b = q_0 a + r_1$. Since

$$\begin{pmatrix} a & r_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & q_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -q_0 \\ 0 & 1 \end{pmatrix},$$

we see that $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is similar to $\begin{pmatrix} a & r_1 \\ 0 & 0 \end{pmatrix}$, and so it suffices to prove that this latter matrix lies in $ID_2(R)$. At the second step we get $a = q_1 r_1 + r_2$. We get the following relation of similarity

$$\begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} \begin{pmatrix} a & r_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q_1 & 1 \end{pmatrix} = \begin{pmatrix} r_2 & r_1 \\ q_1 r_2 & q_1 r_1 \end{pmatrix};$$

moreover

$$\begin{pmatrix} r_2 & r_1 \\ q_1 r_2 & q_1 r_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ q_1 & 0 \end{pmatrix} \begin{pmatrix} r_2 & r_1 \\ 0 & 0 \end{pmatrix}.$$

Hence it suffices to show that $\begin{pmatrix} r_2 & r_1 \\ 0 & 0 \end{pmatrix}$ lies in $ID_2(R)$. Repeating this procedure, after n steps it remains to verify that $\begin{pmatrix} r_n & r_{n-1} \\ 0 & 0 \end{pmatrix} \in ID_2(R)$. Since $r_n = 0$ we get

$$\begin{pmatrix} 0 & r_{n-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & r_{n-1} \\ 0 & 1 \end{pmatrix}.$$

We conclude that $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is similar to a matrix in $ID_2(R)$. □

It is easy to show that a Bézout domain with stable range one admits a weak Euclidean algorithm. Moreover, semi-local domains have stable range one (see [14, V.8.2]). In particular, if $V_1, \ldots, V_n$ are valuation domains of the same field Q, the ring $R = \bigcap_{i=1}^n V_i$ is Bézout and semi-local, hence it admits a weak Euclidean algorithm.

Now we construct some special but quite large classes of infinite intersections of valuation domains that admit a weak Euclidean algorithm.

Proposition 6.3. *Let λ be any cardinal number, and $\{G_\alpha\}_{\alpha<\lambda}$ any family of (not necessarily distinct) totally ordered Abelian groups indexed by λ. Then there exists a family $\{V_\alpha\}_{\alpha<\lambda}$ of valuation domains of the same field Q, such that*

(i) G_α is the value group of V_α;
(ii) $R = \bigcap_{\alpha<\lambda} V_\alpha$ admits a weak Euclidean algorithm.

Proof. For $\alpha < \lambda$, we denote by G_α^+ the subset of the positive elements of the ordered group G_α. Let K be a field, $\{X_\alpha^g\colon\ \alpha < \lambda,\ g \in G_\alpha^+\}$ indeterminates, and consider the polynomial ring $D = K[X_\alpha^g\colon\ \alpha < \lambda,\ g \in G_\alpha^+]$ and its field of quotients $Q = K(X_\alpha^g\colon\ \alpha < \lambda,\ g \in G_\alpha^+)$. For $\alpha < \lambda$, we consider the valuation v_α on Q that extends in the natural way the assignments

$$v_\alpha\big(X_\alpha^g\big) = g, \qquad v_\alpha\big(X_\beta^h\big) = 0, \quad \beta \neq \alpha,\ g \in G_\alpha^+,\ h \in G_\beta^+.$$

Let V_α be the valuation domain of Q associated to v_α; by definition, G_α is the value group of V_α. Let us prove that $R = \bigcap_{\alpha<\lambda} V_\alpha \supset D$ admits a weak Euclidean algorithm. For every finite subset F of λ, we consider the subfield $Q_F = K(X_\gamma^g\colon\ \gamma \in F,\ g \in G_\gamma^+)$ of Q, and the corresponding valuation domains $W_\gamma = V_\gamma \cap Q_F$ $(\gamma \in F)$. Then the ring $R_F = \bigcap_{\gamma\in F} W_\gamma$ admits a weak algorithm, since it is a finite intersection of valuation domains of the same field. Moreover $R_F \subseteq R$ for every finite subset F of λ, since $V_\alpha \cap Q_F = Q_F$ for every $\alpha \notin F$. Indeed, from the definitions it follows that $v_\alpha(f) = 0$ if $f \in Q_F$ and $\alpha \notin F$. Now we pick two arbitrary elements $f, g \in R$; then there exists a suitable finite subset F of λ such that $f, g \in Q_F$, and hence $f, g \in R_F$. Since $R_F \subseteq R$, a weak algorithm satisfied by f, g in R_F is also a weak algorithm in R. The desired conclusion follows. □

The following result provides a natural method for constructing domains with a weak Euclidean algorithm.

Proposition 6.4. *Let D be an integral domain admitting a weak Euclidean algorithm, Q its field of fractions, X an indeterminate. Then the pull-back $R = D + XQ[\![X]\!]$ admits a weak Euclidean algorithm.*

Proof. Let $J = XQ[\![X]\!]$ be the Jacobson radical of R, and pick any $z \in R$. It is well known, and easily verified, that, if $z \notin J$, than $z = au$, for suitable $a \in D$ and $u \in U(R)$, while, if $z \in J$, then $z = qX^k w$, for suitable $q \in Q$, $k > 0$, $w \in U(R)$. Let us pick two arbitrary non-zero elements z_1, z_2 in R, and prove that they satisfy a weak Euclidean algorithm. We distinguish the various possibilities.

Assume that both z_1, z_2 are not in J. Then, thanks to fact (ii) above, we may assume that $z_1, z_2 \in D$, and therefore z_1, z_2 satisfy a weak algorithm in D. That same algorithm works in R, since $D \subset R$.

Assume that $z_1 \notin J$ and $z_2 \in J$, or vice versa. Then one element divides the other, hence z_1, z_2 satisfy a weak algorithm, by fact (i) above.

Assume that $z_1, z_2 \in J$. Again by fact (ii), it is not restrictive to suppose that

$$z_1 = a\big(X^k/d\big), \qquad z_2 = b\big(X^m/d\big),$$

for suitable $a, b, d \in D$, and positive integers k, m. If now $k \neq m$, then one element divides the other, and so z_1, z_2 satisfy a weak algorithm. It remains to examine the case where $k = m$. Under these circumstances, from a weak algorithm for $a, b \in D$ in D, multiplying the relations by $X^k/d \in R$ we derive a weak algorithm in R satisfied by $z_1 = a(X^k/d)$, $z_2 = b(X^k/d)$. □

Remark 2. The classical examples of non-Euclidean PIDs, like special rings of integers in imaginary quadratic number fields or the example constructed by Bass [2], do not satisfy (ID_2). As a matter of fact, they were exhibited for the purpose of showing that not every PID satisfies (GE_2) (cf. [7] and [2]), or, equivalently, (ID_2), by Theorem 3.5. However, somehow surprisingly, an example of a non-Euclidean PID that does satisfy (ID_2) was not found up to now.

It is worth noting that there is no hope to find such example in the natural environment of number fields. In fact, in the imaginary quadratic case, Cohn [7] has proved that a non-Euclidean PID does not satisfy (GE_2). Moreover, Weinberger [26] has proved, under the Generalized Riemann Hypothesis, that every ring of integers R which is a PID is also a Euclidean domain (not necessarily with respect to the usual norm), except for the case where R is imaginary quadratic. More recently, Harper and Murty [16] proved the same result without assuming GRH, but with the additional hypothesis that the unit rank of R is greater than 3.

Actually, O'Meara's Theorem 6.1 ensures that a PID satisfying (ID_2) must admit a weak Euclidean algorithm; hence that result seems to suggest that any non-Euclidean PID cannot satisfy (ID_2).

We end this section proving that a Bézout generalized Euclidean domain is an elementary divisor ring. Recall that a (not necessarily commutative) ring R is said to be an elementary divisor ring (EDR, for short) if the following property is satisfied (cf. [17]).

(∗) For every matrix $\mathbf{M}$ with entries in R there exist invertible matrices $\mathbf{P}$, $\mathbf{Q}$ such that $\mathbf{PMQ} = \operatorname{diag}(d_1 \dots d_n)$ were d_i divides d_{i+1}, for $1 \leqslant i \leqslant n-1$.

Proposition 6.5. *If the Bézout domain R is generalized Euclidean, then it is an EDR.*

Proof. In view of Theorem 5.1 in [17] (and its remark), it suffices to prove that any singular 2×2 matrix $\mathbf{T}$ satisfies property (∗). By O'Meara's Theorem 6.1, R admits a weak Euclidean algorithm. Then the proof of Theorem 6.2 shows that $\mathbf{T}$ is similar to a matrix of the form $\left(\begin{smallmatrix} 0 & r \\ c & d \end{smallmatrix}\right)$, where $rc = 0$, since the matrix is singular. Assume that $r \neq 0$

and $c = 0$. The transpose of this matrix is similar to a matrix of the form $\begin{pmatrix} 0 & 0 \\ 0 & r_1 \end{pmatrix}$, again arguing as in the proof of Theorem 6.2. This last matrix satisfies (∗), since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & r_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that also **T** satisfies (∗). The same argument works when $r = 0$. □

Note that the above proposition is not reversible, since any PID is an EDR, and there exist PIDs that don't satisfy (GE_2), as shown in [7] and [2]. Moreover, not every generalized Euclidean domain is an EDR, since there exist non-Bézout generalized Euclidean domains (e.g., local non-valuation domains), while the finitely generated ideals of an EDR are always principal.

7. Open problems

(1) Determine the domains R such that the two conditions (SC_n) and (SFC_n) are equivalent.

(2) Does condition (SFC_n) imply condition (ID_n) for any integral domain?

(3) Show that a PID admitting a weak Euclidean algorithm is a Euclidean ring, in the classical sense.

The last problem appears to be crucial for the theory, in view of Theorem 3.5.

(4) Show that an integral domain R satisfying property (ID_2) is necessarily a Bézout domain.

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