Combinatorial Identities: Binomial Coefficients, Pascal's Triangle, and Stars & Bars

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Outline:

- Background
  - Factorials
  - Binomial Coefficients
- Pascal's Triangle
  - Several Combinatorial Identities
  - Block Walking Interpretation of the Entries
  - Properties of the Diagonals
- Compositions
  - Famous Combinatorial Identitiy

Background

Factorials: Definition & Examples

For any positive integer n:

$$\boxed{n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}$$

Examples:

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \cdot 1 \\ 3! &= 3 \cdot 2 \cdot 1 \\ \vdots & \quad \vdots \end{aligned}$$

By convention: $0! = 1$

Binomial Coefficients: Definition & Examples

For integers n and k with $0 \leq k \leq n$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples:

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6$$

$$\binom{3}{0} = \frac{3!}{0!3!} = \frac{6}{1 \cdot 6} = 1$$

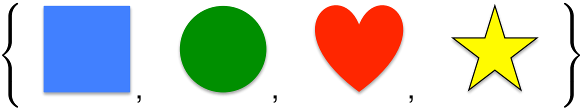
Binomial Coefficients: Interpretation

$\binom{n}{k}$ = the # of ways to choose a subset with k elements from a set with n elements

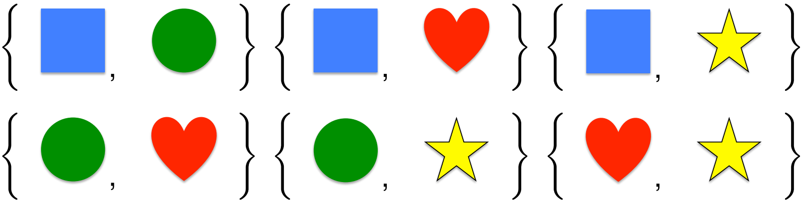
Equivalently:

$\binom{n}{k}$ = the # of k-element subsets that can be formed from a set with n elements

Binomial Coefficients: An Example

$S =$ 

Subsets of S with 2 elements:



$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6 = \#\text{ of subsets of } S \text{ with 2 elements } \checkmark$$

Pascal’s Triangle

Pascal’s Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\iddots \qquad \qquad \ddots$$

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdots \quad \binom{n}{n-2} \quad \binom{n}{n-1} \quad \binom{n}{n}$$

Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
 \udots & & & & & & & & & & & & \ddots \\
1 & & n & & \binom{n}{2} & & \cdots & & \binom{n}{n-2} & & n & & 1
\end{array}
$$

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Our First Identity

Pascal's Triangle: Sums of Adjacent Terms

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Adjacent Terms

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Adjacent Terms

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal’s Triangle: Sums of Adjacent Terms

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Adjacent Terms

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal’s Triangle: Sums of Adjacent Terms

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
⋮ ⋮

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 **35** **35** 21 7 1

1 8 28 56 **70** 56 28 8 1

$\vdots$ $\vdots$

$\cdots \quad \boxed{\binom{n-1}{k-1}} \quad \boxed{\binom{n-1}{k}} \quad \cdots$

$\boxed{\binom{n}{k}}$

Identity

$$\boxed{\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}}$$

Combinatorial Proof

Consider the set $S = \{1, 2, 3, 4, \ldots, n\}$

Plan: Count the # of subsets of S with k elements in two different ways.

Combinatorial Proof

1st way to count the # of subsets of S with k elements:

$$\text{Total } \# \text{ of } k\text{-element subsets of } S = \binom{n}{k}$$

Combinatorial Proof

2nd way to count the # of subsets of S with k elements:

Observation: Each subset of S either contains the element n or it doesn't.

Total # of k-element subsets of S	$=$	# of k-element subsets of S that contain n	$+$	# of k-element subsets of S that don't contain n

Combinatorial Proof

To form a k-element subset that contains n:

- Put n into the subset. There's only 1 way to do this.
- Choose $k - 1$ other elements from $\{1, 2, 3, \ldots, n - 1\}$
  There are $\binom{n-1}{k-1}$ ways to do this.

$\Longrightarrow$ There are $\binom{n-1}{k-1}$ k-element subsets that contain n.

To form a k-element subset that *doesn't* contain n:

- Choose k elements from $\{1, 2, 3, \ldots, n - 1\}$
  There are $\binom{n-1}{k}$ ways to do this.

$\Longrightarrow$ There are $\binom{n-1}{k}$ k-element subsets that *don't* contain n.

Combinatorial Proof

$$\begin{array}{c}\text{Total } \# \text{ of } k\text{-element} \\ \text{subsets of } S\end{array} = \begin{array}{c}\# \text{ of } k\text{-element subsets} \\ \text{of } S \text{ that contain } n\end{array} + \begin{array}{c}\# \text{ of } k\text{-element subsets} \\ \text{of } S \text{ that don't contain } n\end{array}$$

$$\begin{array}{c}\text{Total } \# \text{ of } k\text{-element} \\ \text{subsets of } S\end{array} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

In Summary:

1st way: Total # of k-element subsets of S $= \binom{n}{k}$

2nd way: Total # of k-element subsets of S $= \binom{n-1}{k-1} + \binom{n-1}{k}$

$$\Longrightarrow \boxed{\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}} \qquad \square$$

Our Second Identity

Pascal's Triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1
⋮														⋮

Pascal's Triangle: The Sums of the Rows

																		Sum
								1										1
							1		1									2
						1		2		1								4
					1		3		3		1							8
				1		4		6		4		1						16
			1		5		10		10		5		1					32
		1		6		15		20		15		6		1				64
	1		7		21		35		35		21		7		1			128
	⋮														⋮			

Pascal's Triangle: The Sums of the Rows

							1									$1 = 2^0$
						1		1								$2 = 2^1$
					1		2		1							$4 = 2^2$
				1		3		3		1						$8 = 2^3$
			1		4		6		4		1					$16 = 2^4$
		1		5		10		10		5		1				$32 = 2^5$
	1		6		15		20		15		6		1			$64 = 2^6$
1		7		21		35		35		21		7		1		$128 = 2^7$
$\vdots$														$\vdots$		

$\binom{n}{0}$ $\binom{n}{1}$ $\binom{n}{2}$ $\binom{n}{3}$ $\cdots$ $\binom{n}{n-3}$ $\binom{n}{n-2}$ $\binom{n}{n-1}$ $\binom{n}{n}$ 2^n

Another Identity

$$\boxed{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n} = 2^n}$$

Combinatorial Proof:

Again, consider the set $S = \{1, 2, 3, \ldots, n\}$.

Plan: Count the total number of subsets of S in two different ways.

Combinatorial Proof:

1st way of counting the subsets of S:

# of subsets w/ no elements	+	# of subsets w/ 1 element	+	# of subsets w/ 2 elements	$+ \cdots +$	# of subsets w/ n-1 elements	+	# of subsets w/ n elements
$\binom{n}{0}$	$+$	$\binom{n}{1}$	$+$	$\binom{n}{2}$	$+ \cdots +$	$\binom{n}{n-1}$	$+$	$\binom{n}{n}$

Combinatorial Proof:

2nd way of counting the subsets of S:

When building a subset of S, there are two choices for each element: either it's in the subset or it's not.

$\implies$ There are $2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2 = 2^n$ ways to build a subset.

$\implies$ There are 2^n subsets of S.

In Summary:

1st Way:

$$\text{Total \# of subsets of } S = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

2nd Way:

$$\text{Total \# of subsets of } S = 2^n$$

$$\therefore \boxed{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n}$$

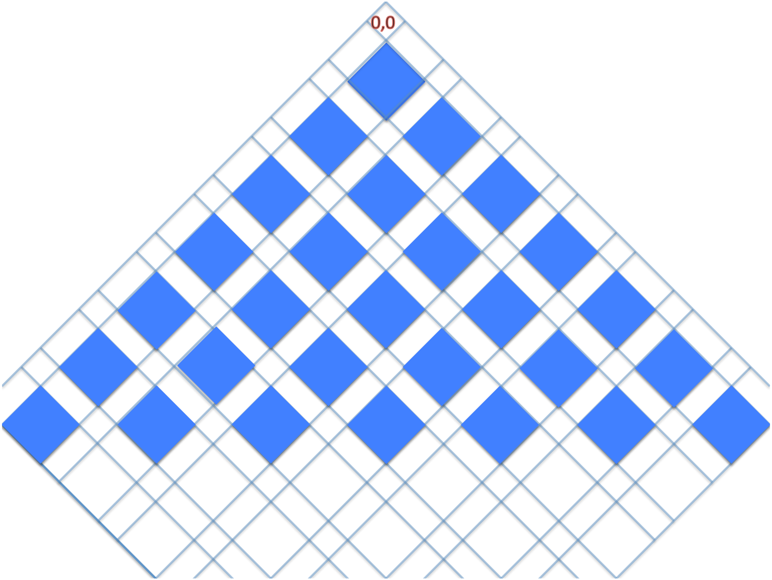
□

Block Walking

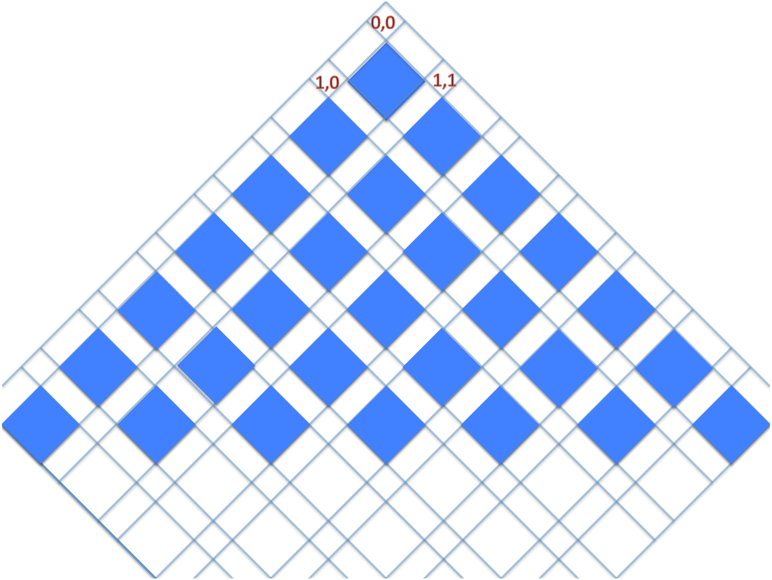
An interpretation of the binomial coefficients

Block walking: The grid of blocks

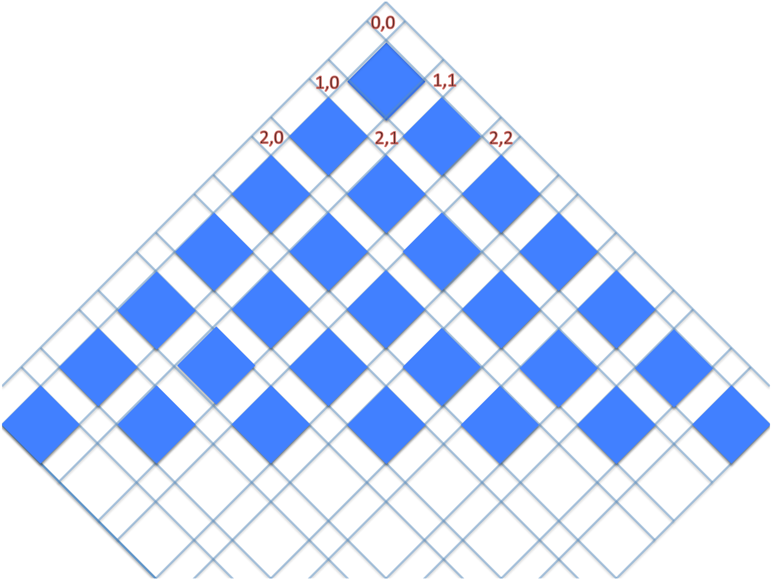
Block walking: Coordinates



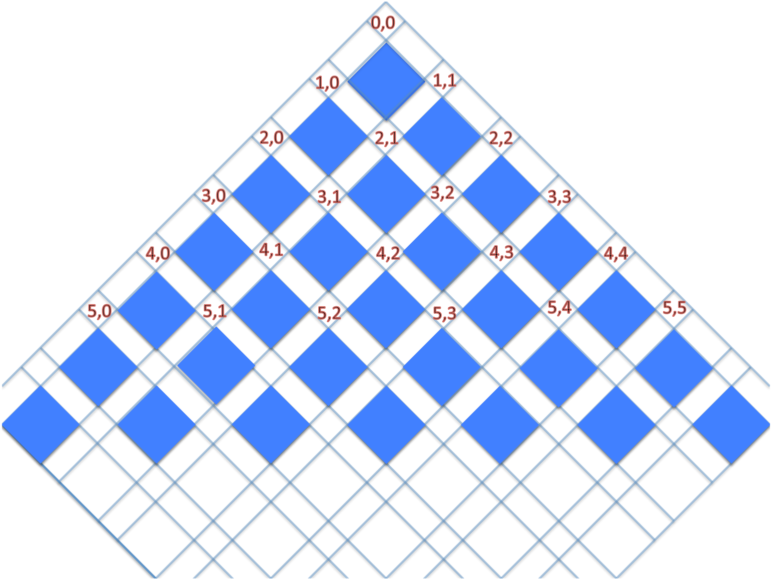
Block walking: Coordinates



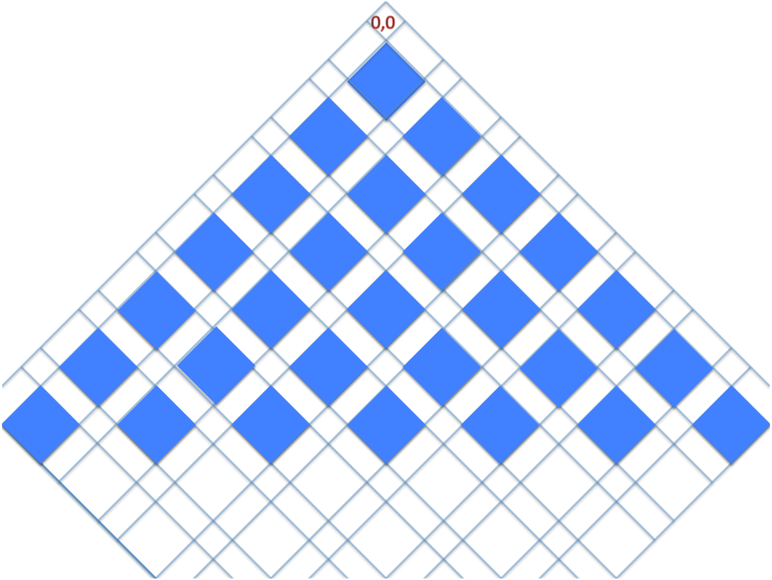
Block walking: Coordinates



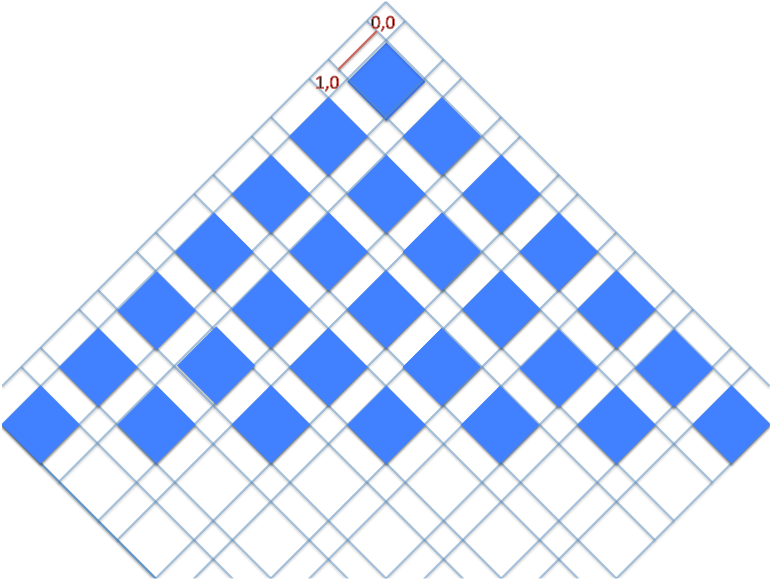
Block walking: Coordinates



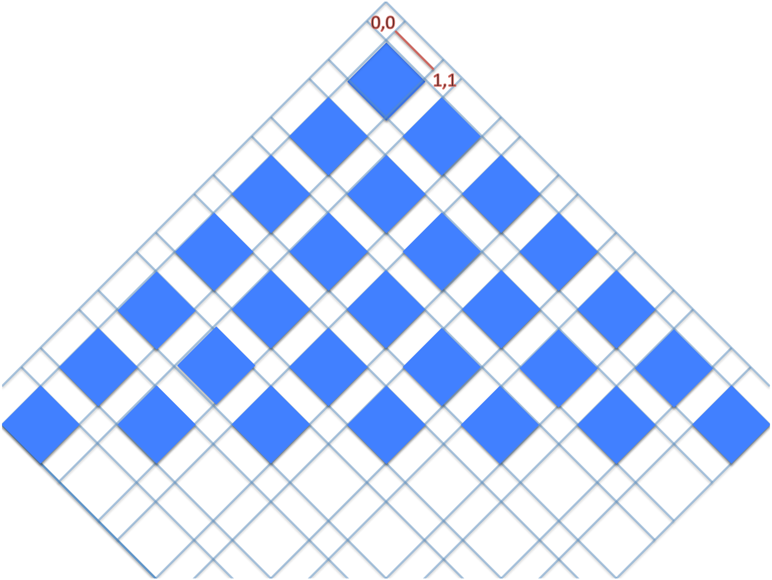
Block walking: The number of walks from 0,0 to 0,0



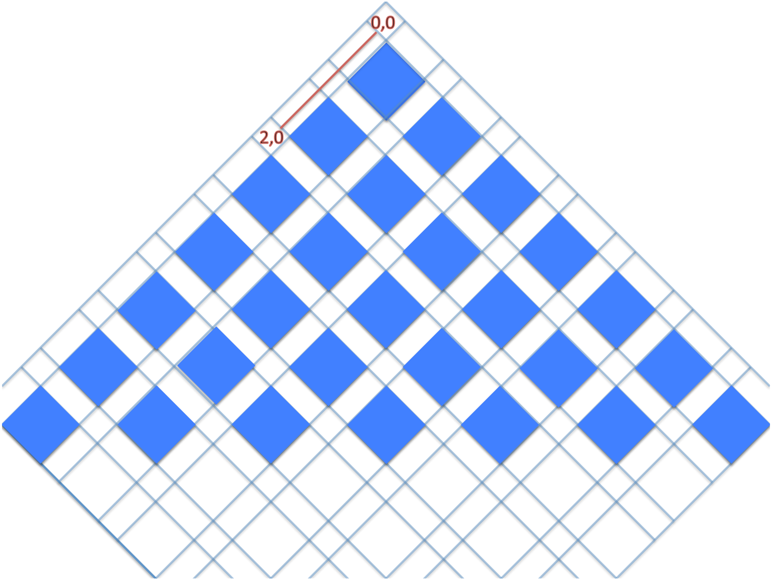
Block walking: The number of walks from 0,0 to 1,0



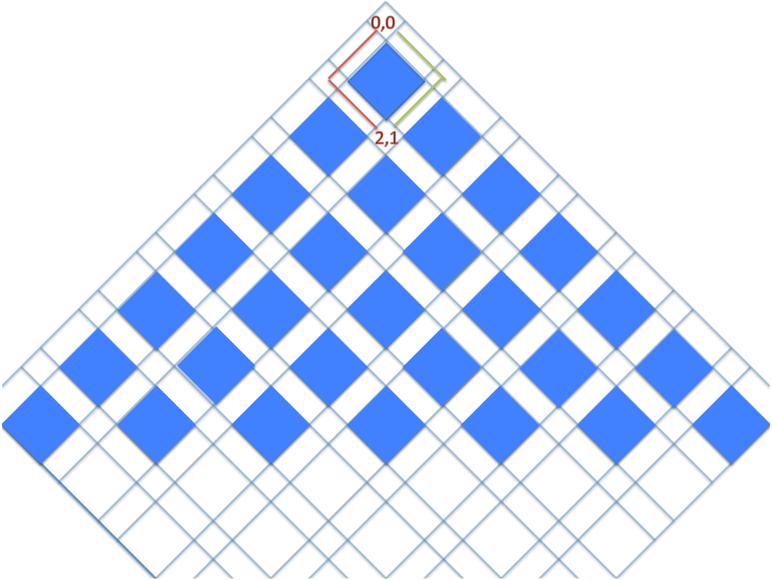
Block walking: The number of walks from 0,0 to 1,1



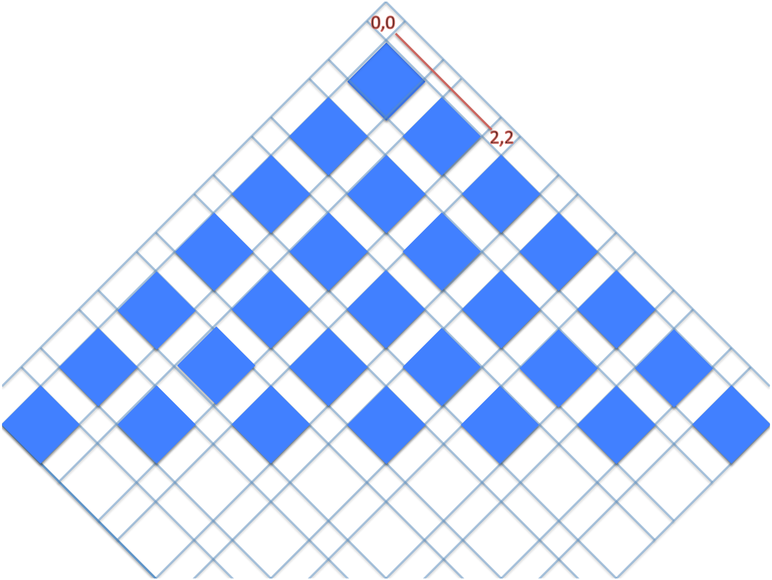
Block walking: The number of walks from 0,0 to 2,0



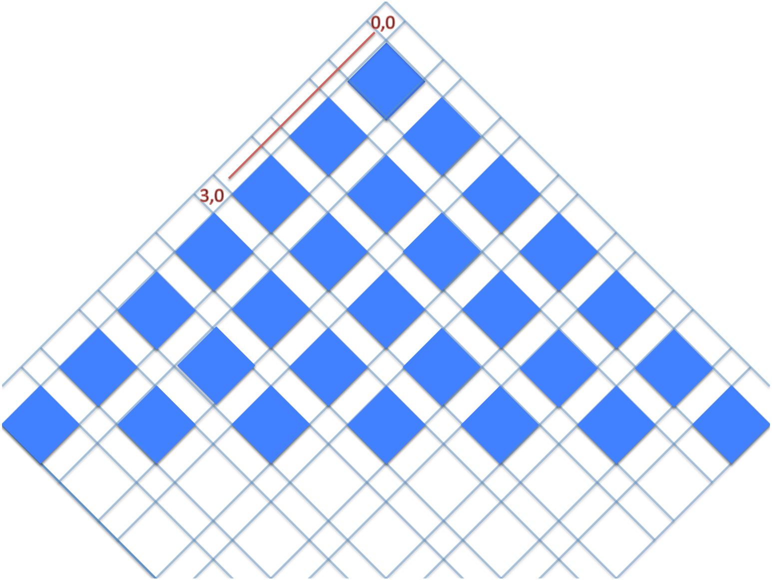
Block walking: The number of walks from 0,0 to 2,1



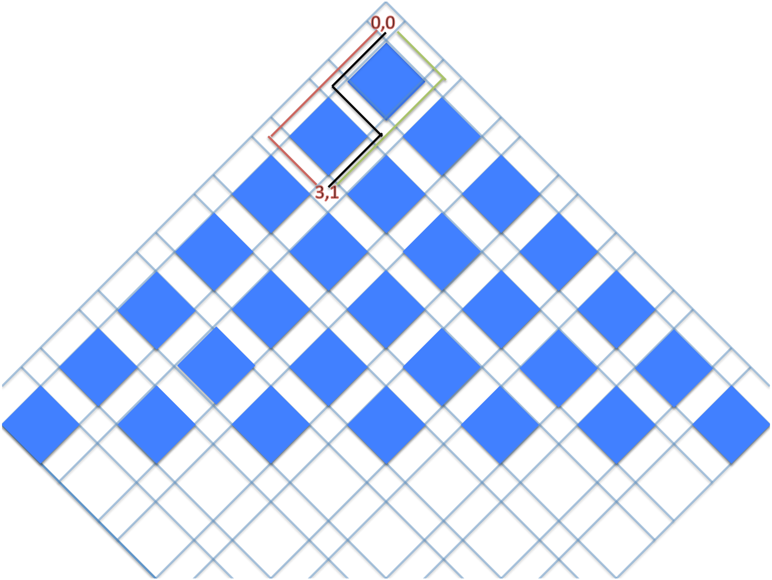
Block walking: The number of walks from 0,0 to 2,2



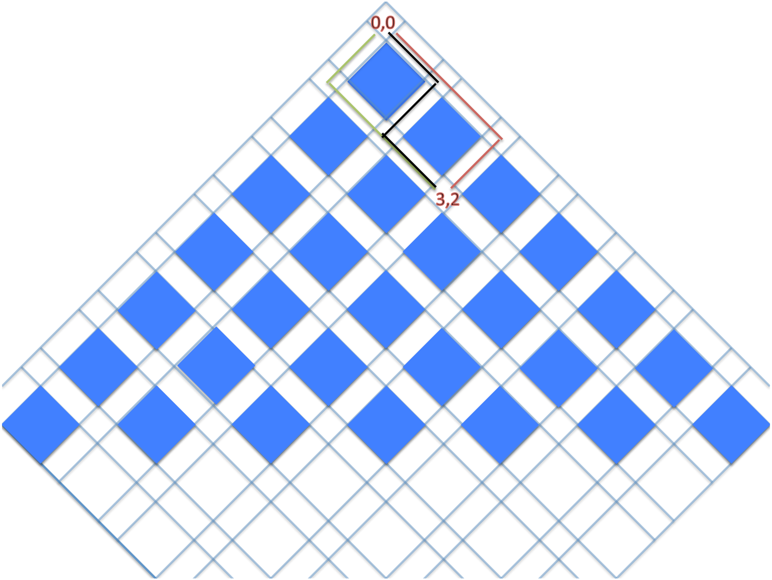
Block walking: The number of walks from 0,0 to 3,0



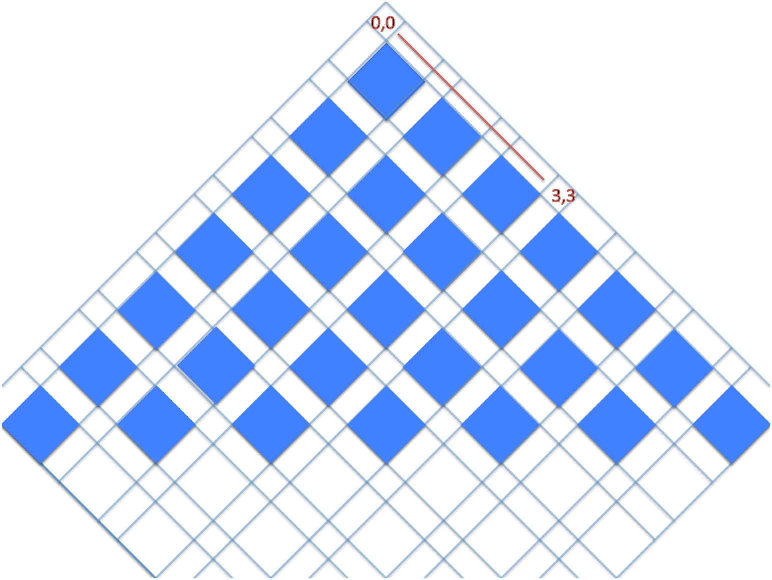
Block walking: The number of walks from 0,0 to 3,1



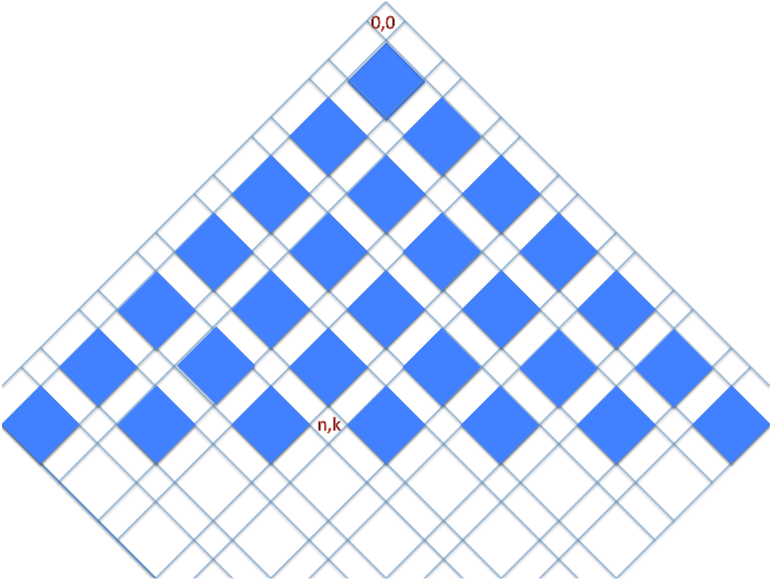
Block walking: The number of walks from 0,0 to 3,2



Block walking: The number of walks from 0,0 to 3,3



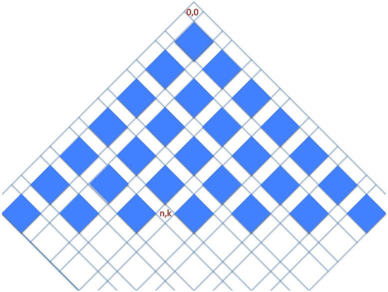
Block walking: The number of walks from 0,0 to n,k



Block walking: The number of walks from 0,0 to n,k

The # of block walks from 0,0 to n,k is $\binom{n}{k}$

Combinatorial Proof:



To walk from 0,0 to n,k:

- You need to travel n total blocks to get to the n^{th} row.
- k of these n blocks must be to the right to end up in the k^{th} position in the row.

# of block walks from 0,0 to n,k: $\binom{n}{k}$

Our 3rd Identity

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal’s Triangle: Sums of Diagonals

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Diagonals

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Diagonals

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1

⋮ ⋮

Pascal's Triangle: Sums of Diagonals

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Diagonals

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Diagonals

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1
⋮																⋮

Pascal's Triangle: Sums of Diagonals

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Diagonals

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Diagonals

1

1 1

1 2 1

1 3 3 1

[1] 4 6 4 1

1 [5] 10 10 5 1

1 6 [15] 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Diagonals

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Pascal's Triangle: Sums of Diagonals

$\binom{n}{0}$

$\binom{n+1}{1}$

$\binom{n+2}{2}$

$\binom{n+3}{3}$

$\ddots$

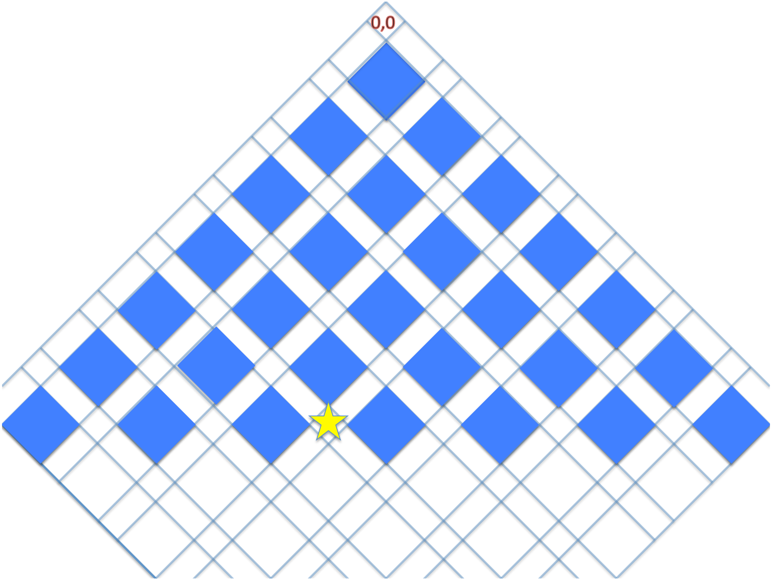
$\binom{n+r}{r}$

$\binom{n+r+1}{r}$

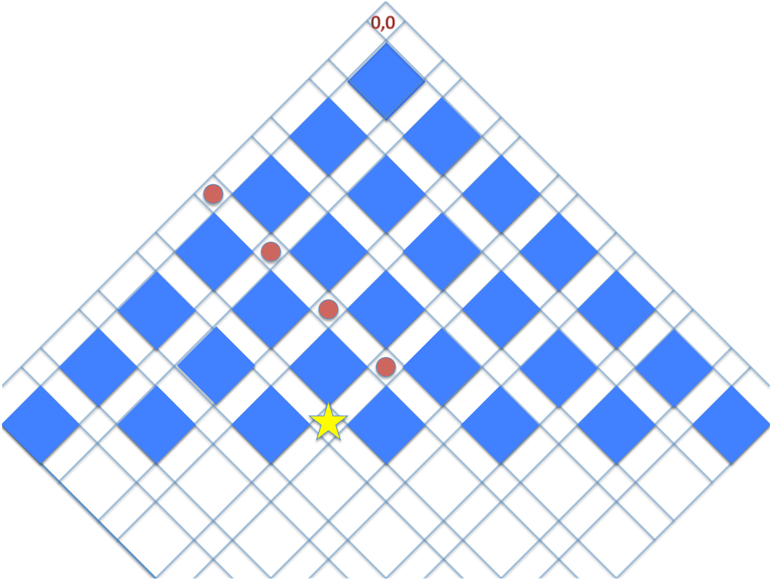
Identity

$$\boxed{\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\binom{n+3}{3}+\cdots+\binom{n+r}{r}=\binom{n+r+1}{r}}$$

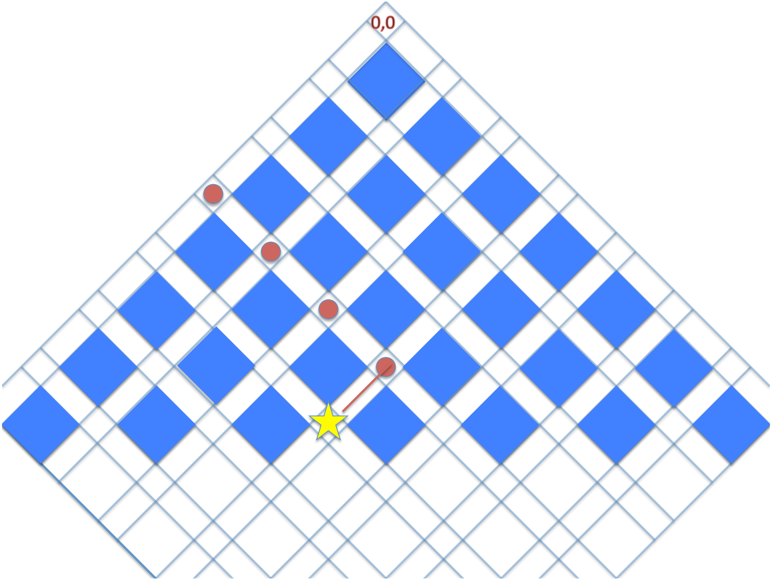
Combinatorial Proof: Block Walks!



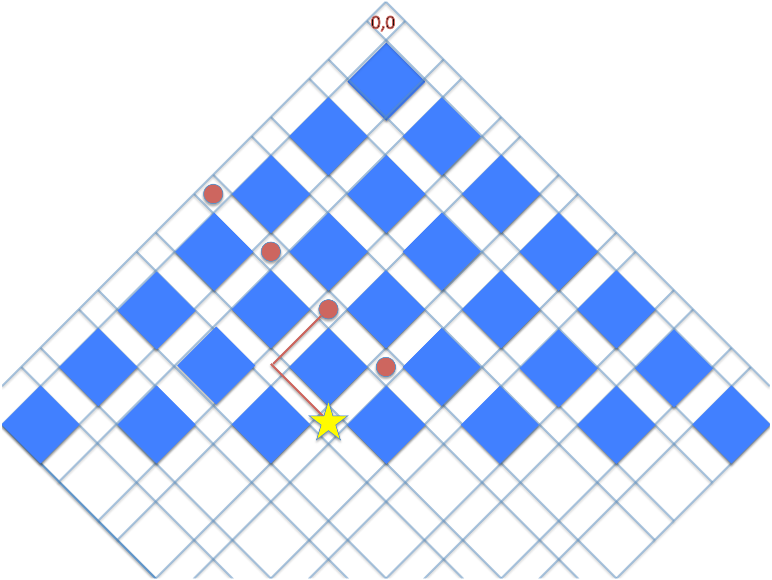
Find the last possible left turns:



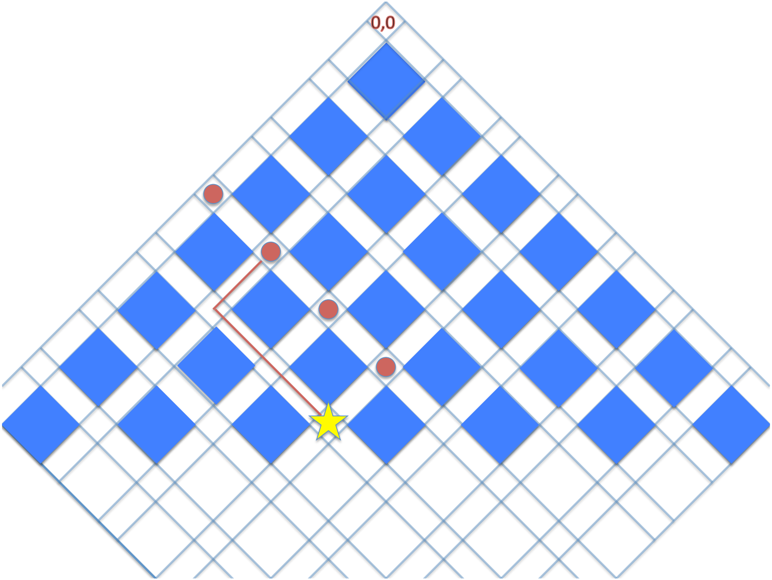
Find the last possible left turns:



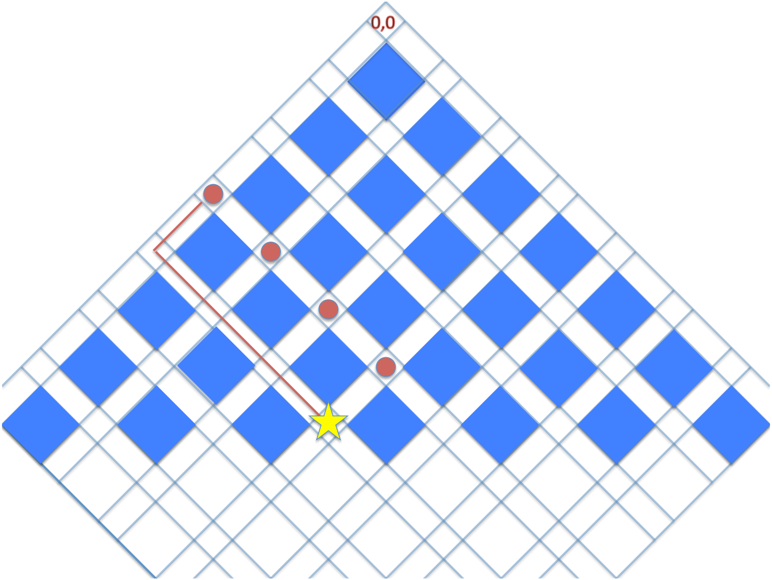
Find the last possible left turns:



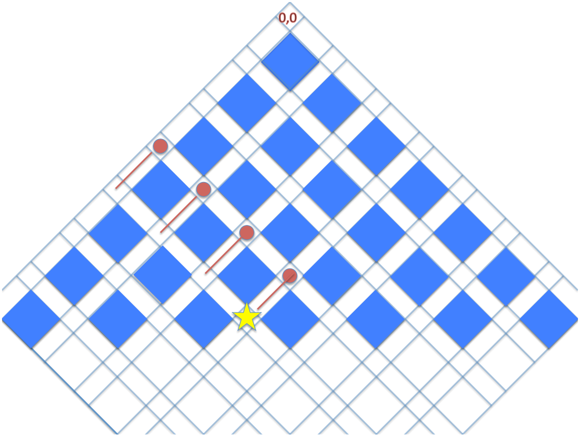
Find the last possible left turns:



Find the last possible left turns:



Key Observation:



From each of the last left turns, there's only one way to get to ☆.

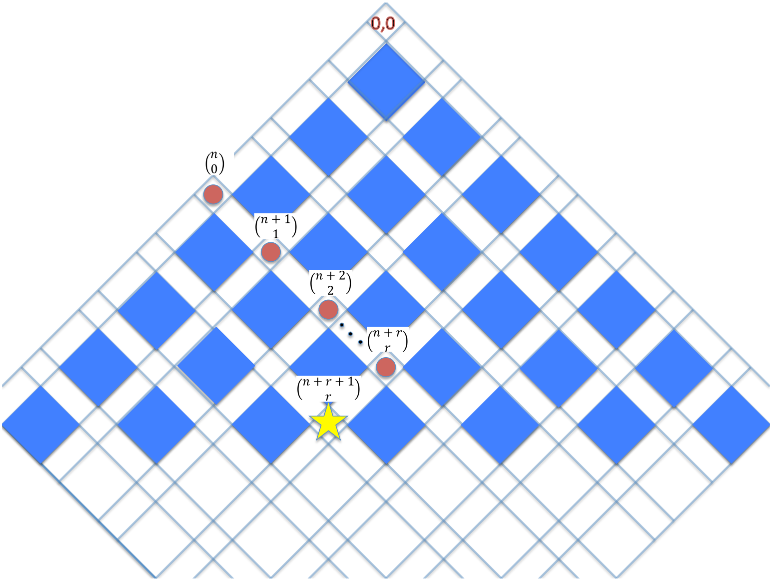
Counting the Block Walks

# of block walks to ★	=	# of block walks whose last left turn is at the 1st circle
	+	# of block walks whose last left turn is at the 2nd circle
	⋮	
	+	# of block walks whose last left turn is at the last circle

# of block walks to ★	=	# of block walks from 0,0 to the 1st circle
	+	# of block walks from 0,0 to the 2nd circle
	⋮	
	+	# of block walks from 0,0 to the last circle

Using the Interpretation of the Binomial Coefficients:

For some *n*:



Counting the Block Walks:

# of block walks to ☆	=	# of block walks from 0,0 to the 1st circle
	+	# of block walks from 0,0 to the 2nd circle
	⋮	
	+	# of block walks from 0,0 to the last circle

$$\binom{n+r+1}{r} = \binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+r}{r}$$

□

Other Diagonals

Pascal's Triangle: The 3rd Diagonal

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Triangular Numbers!

Triangular Numbers

Idea: Form equilateral triangles using equally spaced dots stacked on top of each other.

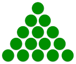
T_n = the number of dots in such a triangle with side length of n.

$T_1 = 1$

$T_2 = 3$

$T_3 = 6$

$T_4 = 10$



$T_5 = 15$

Pascal's Triangle: The 3rd Diagonal

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1
⋮																⋮

Triangular Numbers!

Pascal's Triangle: The 4th Diagonal

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Tetrahedral Numbers!

Tetrahedral Numbers

Idea: Form regular tetrahedrons using equally spaced balls stacked on top of each other.

H_n = the number of balls in such a tetrahedron with side length of n.

1: 1 ball $\implies H_1 = 1$

2: 3 balls in the base
1 ball on top $\implies H_2 = 4$

3: 6 balls in the base
3 balls in the middle layer
1 ball on top $\implies H_3 = 10$

4: 10 balls in the base
6 balls in the 2nd layer
3 in the 3rd layer
1 ball on top $\implies T_4 = 20$

Pascal's Triangle: The 4th Diagonal

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

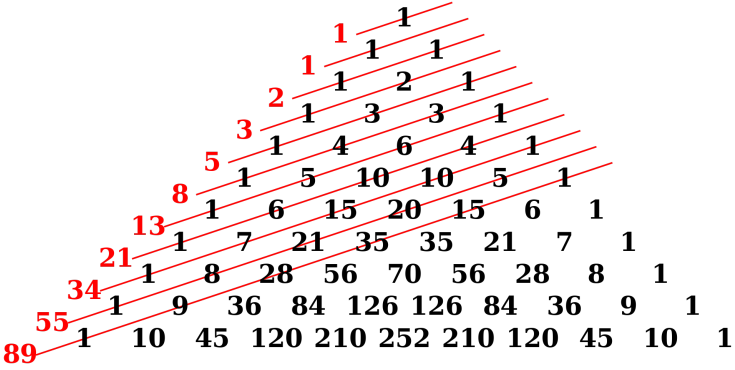
1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

⋮ ⋮

Tetrahedral Numbers!

Pascal’s Triangle: Shallow Diagonals



Source: http://en.wikipedia.org/wiki/File:PascalTriangleFibanacci.svg Author: RDBury

Fibonacci Numbers!

Counting Compositions

Counting Compositions

Let n be a positive integer.

A *composition* of n: An ordered sum of positive integers that add up to n.

Example: There are eight compositions of 4:

$$1+1+1+1=4 \qquad 2+2=4$$

$$1+1+2=4 \qquad 1+3=4$$

$$1+2+1=4 \qquad 3+1=4$$

$$2+1+1=4 \qquad 4=4$$

The Eight Compositions of 4:

- # of Compositions w/ Four Terms: 1
  - 1+1+1+1=4
- # of Compositions w/ Three Terms: 3
  - 1+1+2=4
  - 1+2+1=4
  - 2+1+1=4
- # of Compositions w/ Two Terms: 3
  - 2+2=4
  - 1+3=4
  - 3+1=4
- # of Compositions w/ One Term: 1
  - 4=4

The Eight Compositions of 4:

- # of Compositions w/ Four Terms: $1 = \binom{3}{3}$
  - 1+1+1+1=4
- # of Compositions w/ Three Terms: $3 = \binom{3}{2}$
  - 1+1+2=4
  - 1+2+1=4
  - 2+1+1=4
- # of Compositions w/ Two Terms: $3 = \binom{3}{1}$
  - 2+2=4
  - 1+3=4
  - 3+1=4
- # of Compositions w/ One Term: $1 = \binom{3}{0}$
  - 4=4

Identity

$$\text{\# of compositions of } n \text{ with } k \text{ terms} = \binom{n-1}{k-1}$$

Combinatorial “Stars & Bars” Proof

Start with the stars:

Place n stars in a row: $\star \; \star \; \star \; \star \; \star \cdots \star \; \star \; \star$

Then add the bars:

Place $k - 1$ bars in the $n - 1$ spaces between the n stars (w/ at most 1 bar in each space):

$$\star \,|\, \star \; \star \; \star \,|\, \star \; \cdots \,|\, \star \,|\, \star \; \star$$

Key Observation

Counting the # of stars between the bars:

$$\text{each placement of bars} \quad \longleftrightarrow \quad \text{a composition of } n$$

$$\star | \star \star \star | \star \cdots | \star | \star \star \quad \longleftrightarrow \quad 1 + 3 + \cdots + 1 + 2 = n$$

Key Observation

There's a (bijective) correspondence between the sets:

$$\left\{ \begin{array}{c} \text{Placements of } k-1 \text{ bars} \\ \text{in the } n-1 \text{ spaces} \\ \text{between the } n \text{ stars} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Compositions} \\ \text{of } n \text{ with} \\ k \text{ terms} \end{array} \right\}$$

Example:

Placements of 2 bars in the 3 spaces between the 4 stars

$$\star | \star | \star \; \star \quad \longleftrightarrow \quad 1+1+2=4$$

$$\star | \star \; \star | \star \quad \longleftrightarrow \quad 1+2+1=4$$

$$\star \; \star | \star | \star \quad \longleftrightarrow \quad 2+1+1=4$$

Compositions of 4 w/ 3 terms

A Consequence of the Correspondence

# of placements of $k-1$ bars in the $n-1$ spaces between the n stars (with at most 1 bar in each space)	=	**# of** compositions of n with k terms

Counting the # of Placements

# of placements of $k-1$ bars in the $n-1$ spaces between the n stars (with at most 1 bar in each space)	=	# of ways to choose $k-1$ objects from a set of $n-1$ objects
	=	$\binom{n-1}{k-1}$

In Summary:

# of compositions of n with k terms	=	# of placements of $k-1$ bars in the $n-1$ spaces between the n stars (with at most 1 bar in each space)

# of placements of $k-1$ bars in the $n-1$ spaces between the n stars (with at most 1 bar in each space)	=	$\binom{n-1}{k-1}$

$\Longrightarrow$ **# of compositions of n with k terms** $= \binom{n-1}{k-1}$

$\square$